MANAGING SOLVENCY: A RISK THEORY INSIGHT

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Abstract. In solvency testing, it is always advisably to precede enhanced modelling and application of DFA methodology by a risk theory insight. Addressing a simplified model, the author puts forth an adaptive control strategy designed to balance the requirements of the principles of solvency and equity in a many-year perspective.

Introduction. Keystone methodology flourished recently under the name of Dynamic Financial Analysis (DFA), relies upon the idea of Pentikäinen to enhance modelling of the insurance process and to apply simulation for different strategies. It is generally accepted that the modern DFA is neither an academic discipline, nor a single economic or mathematical concept or method. It may be described as modelling an entire insurance process on a cash flow basis by a comparison of different management strategies and economic scenarios in terms of risk and return. This comparison is fulfilled by means of simulation.

It is noteworthy that accentuating the demands of practical management, Pentikäinen voted for a compromise between simulation and analytical methods. One great advantage of the analytic method, even if it is based on very special assumptions — wrote Pentikäinen — is that the interdependence of the variables involved can be illustrated. Even if the values obtained are far from the values obtained by the original assumptions, probably at least the shape of the interdependence can be preliminary studied in this way which makes it easier to understand the structure of the complicated model. How far these analytical results are valid also as a solution of the original problem, which is based on more general and more realistic assumptions, probably cannot be estimated. However, the decision variables obtained analytically can serve initial variables for simulation and the problem of optimization by means of simulation can, perhaps, be made easier.

Although practising actuaries may be frustrated by complicated analytical approach, it is tempting to catch more than a mere "numerical fish" with a fishing net of analytical methods.

A thing before introducing a model further in the paper, is the prerequisite from insurance regulation. Next is a sketch of a general control model of a multiperiodic insurance process. Then we address to synthesis of a zone-adaptive strategy and to its performance, including investment power. The article finishes with a short discussion.

Some prerequisite. Article 16(a) of Directive [**1**] yields an algorithm for calculation of mandatory annual reserves. It is designed to keep the multiperiodic insurance process inside a strip zone within a series of consecutive years. One believes that it makes the development of the insurance process stable and complying with fundamental principles of solvency and equity.

The interest to new solvency and supervisory standards for insurance companies, to be crystallized into Solvency II, is backed by intensive investigations (see, e.g. [**2**], [**3**], [**4**]). In particular, the IAIS document "Principles on Capital Adequacy and Solvency" [**3**] specifies a set of regulatory principles.

Principle 7 of [**3**] claims that a control level is required: "insurance regulatory authorities have to establish a control level, or a series of control levels, that trigger intervention by the authority in an insurer's affairs when the available solvency falls below this control level. These control levels may be supported by a specific framework or by a more general framework providing the supervisor a latitude of action." Principle 8 of [**3**] claims that a minimum level of capital has to be specified, and "the regulatory framework has to set out a threshold minimum capital requirement for companies."

Seeking for duly established control and minimum capital levels, it is generally accepted that the insurance regulation and supervision go blind without a comprehensive mathematical model, or a set of models, which describe, inter alia, how the company might collapse.

A control model of multiperiod insurance process. Though optimization, e.g., optimal pay-out of dividends, is a traditional control problem of actuarial mathematics, long-term steady business is the ultimate goal of insurance management. The theoretical implement to achieve it is the *adaptive control* in many-year models of the insurance process, based on the achievements of the risk theory.

A sensible way to amalgamate the ideas of the adaptive control and collective risk theories is to address to multiperiodic control model composed of chained singleperiodic risk models. A trajectory of the insurance process with annual accounting and subsequent annual control may be diagramed as

$$\mathfrak{w}_0 \underbrace{\xrightarrow{\gamma_0} \mathfrak{u}_0 \xrightarrow{\pi_1} \mathfrak{w}_1}_{\text{1-st year}} \cdots \xrightarrow{\pi_{k-1}} \mathfrak{w}_{k-1} \underbrace{\xrightarrow{\gamma_{k-1}} \mathfrak{u}_{k-1} \xrightarrow{\pi_k} \mathfrak{w}_k}_{k\text{-th year}} \cdots .$$

According to this diagram (for $k = 1, 2, \dots$), at the end of $(k-1)$-th year the state variable $\mathfrak{w}_{k-1}$ assuming values in a space W, is observed. It describes the insurer's position at that moment. Then, at the beginning of k-th year the control rule γ_{k-1} is applied to choose the control variable $\mathfrak{u}_{k-1}$, assuming values in a space U. Thereupon the k-th year probability mechanism of insurance unfolds; the transition function of this mechanism is denoted by π_k. It defines the insurer's position at the end of the k-th year. Particular choice of the elements of the model, including the spaces U and W, depends on the context.

A zone-adaptive control strategy. In this paper the annual probability mechanisms of insurance are Poisson–Exponential, or classical, and diffusion. Both are well known key models of the risk theory.

(1) *Poisson–Exponential, or classical, model*: the risk reserve at time t is

$$R_t(u,\tau) = u + (1+\tau)\frac{\lambda}{\mu}t - V(t), \quad V(t) = \sum_{i=1}^{N(t)} Y_i, \quad t \geqslant 0,$$

where u is the initial risk reserve, τ is the adaptive premium loading, $\{T_i\}_{i\geqslant 1}$ and $\{Y_i\}_{i\geqslant 1}$ are i.i.d. and mutually independent, where T_i are the interclaim times and Y_i are the amounts of claims, exponentially distributed with parameters $\lambda > 0$ and $\mu > 0$, respectively, $N(t)$ is the largest n for which $\sum_{i=1}^{n} T_i \leqslant t$ (we put $N(t) = 0$ if $T_1 > t$). Note that $\mathsf{E}V(t) = \frac{\lambda}{\mu}t$, so that the premium rate is calculated according to the expected value principle, and $\mathsf{D}V(t) = \frac{2\lambda}{\mu^2}t$.

(2) *Diffusion model*: the risk reserve at time t is

$$R_t(u,\tau) = u + (1+\tau)\mu t - V(t), \quad V(t) = \mu t + \sigma \mathsf{W}_t, \quad t \geqslant 0,$$

where u is the initial risk reserve, τ is the adaptive premium loading, W_t is a standard Brownian motion, μ is the premium rate calculated according to the expected value principle, i.e., $\mathsf{E}V(t) = \mu t$, and $\sigma > 0$ is a constant diffusion coefficient, $\mathsf{D}V(t) = \sigma^2 t$.

Assume that the duration of the incoming year is t. Introduce the key notion of *target capital value* of the risk reserve corresponding to a level $0 < \alpha < 1$, denoted by $u_{\mathrm{targ}}(\alpha, t)$. It is a positive solution of the equation

$$\psi_t(u;0) = \mathsf{P}\{\inf_{0<s\leqslant t} R_s(u,0) < 0\} = \alpha. \tag{1}$$

The equation (1) may be called "neutral–loading" or "equitable–reserving". It defines the initial risk reserve sufficient to make the probability of ruin equal to α without resort to premium loading, which may be voted fair by customers.

Bearing in mind the algorithm of Directive [**1**], which forwards the risk reserve inside a strip zone by means of adjusting the initial capital, address additionally to the premium loading. The idea is to have the premium loading monotone increasing, as the risk reserve passes below an upper level (for simplicity, it will be taken equal to $u_{\mathrm{targ}}(\alpha, t)$), but does not exceed a lower alarm level (it will be taken from further solvency considerations). If the lower alarm level is down-crossed, a deficient capital is borrowed. If the upper level is up-crossed, an excessive capital is reserved. The positive balance of the reserved–borrowed capital constitutes the investment power of a zone-adaptive strategy.

Let z be a deviation, either positive or negative, of the past-year-end risk reserve from $u_{\mathrm{targ}}(\alpha, t)$. Case $z < 0$ means deficit under $u_{\mathrm{targ}}(\alpha, t)$, case $z > 0$ means surplus over $u_{\mathrm{targ}}(\alpha, t)$. We refer to

$$u_{z,t} = u_{\mathrm{targ}}(\alpha, t) + z \quad \text{and} \quad \tau_{z,t} = -\frac{z}{\mathsf{E}V(t)}, \quad z \in \mathsf{R},$$

as *basic adaptive strategy*.

It is noteworthy that $R_t(u_{z,t}, \tau_{z,t}) = u_{\mathrm{targ}}(\alpha, t) + \frac{\lambda}{\mu}\,t - \sum_{i=1}^{N(t)} Y_i$ in the classical risk model and $R_t(u_{z,t}; \tau_{z,t}) = u_{\mathrm{targ}}(\alpha, t) - \sigma \mathsf{W}_t$ in the diffusion risk model, so that in both cases

$$\mathsf{E}R_t(u_{z,t}, \tau_{z,t}) = u_{\mathrm{targ}}(\alpha, t) \quad \text{for any } z \in \mathsf{R},$$

which means that the basic adaptive strategy makes the capital at the time t (i.e., at the year-end of a single period) equal "in the average" to the target value $u_{\mathrm{targ}}(\alpha, t)$. That observation justifies the name of the target capital value.

Introduce the notion of *lower alarm level* of a zone with target value $u_{\mathrm{targ}}(\alpha, t)$ and with prescribed level β of the probability of ruin, $0 < \alpha < \beta < 1$. It is

$$u_{\mathrm{low}}(\alpha, \beta, t) = u_{\mathrm{targ}}(\alpha, t) + z(\alpha, \beta, t),$$

where $z(\alpha, \beta, t) < 0$ is a solution of the equation

$$\boldsymbol{\psi}_t(u_{z,t}; \tau_{z,t}) = \mathsf{P}\Big\{ \inf_{0<s\leqslant t} R_s(u_{z,t}, \tau_{z,t}) < 0 \Big\} = \beta. \tag{2}$$

One may prove that in the diffusion model, for $0 < \alpha < 1$, the solution of equation (1) may be written as

$$u_{\mathrm{targ}}(\alpha, t) = \sigma \sqrt{t}\, c_\alpha,$$

where $c_\alpha = \Phi^{-1}_{\{0,1\}}(1 - \alpha/2)$ and, for $0 < \alpha < \beta < 1$, the solution of equation (2) may be written as

$$z(\alpha, \beta, t) = -\sigma \sqrt{t}\, x_{\alpha,\beta},$$

where $x_{\alpha,\beta} > 0$ is a unique root of the equation

$$1 - \Phi_{\{0,1\}}(c_\alpha) + \exp\{-2x(c_\alpha - x)\}\Phi_{\{0,1\}}(2x - c_\alpha) = \beta. \tag{3}$$

TABLE. Values of $x_{\alpha,\beta}$ calculated numerically using (3).

		$\beta = 110\alpha\%$	$\beta = 120\alpha\%$	$\beta = 130\alpha\%$	$\beta = 140\alpha\%$
$\alpha = 0.1$	$c_\alpha = 1.645$	$x_{\alpha,\beta} = 0.7121$	$x_{\alpha,\beta} = 0.7522$	$x_{\alpha,\beta} = 0.7897$	$x_{\alpha,\beta} = 0.8249$
$\alpha = 0.05$	$c_\alpha = 1.960$	$x_{\alpha,\beta} = 0.8360$	$x_{\alpha,\beta} = 0.8736$	$x_{\alpha,\beta} = 0.9090$	$x_{\alpha,\beta} = 0.9422$
$\alpha = 0.01$	$c_\alpha = 2.576$	$x_{\alpha,\beta} = 1.0754$	$x_{\alpha,\beta} = 1.1098$	$x_{\alpha,\beta} = 1.1423$	$x_{\alpha,\beta} = 1.1730$

The solutions $u_{\mathrm{targ}}(\alpha, t)$ and $z(\alpha, \beta, t)$ of equations (1) and (2) in the classical risk model are much more complicated, remaining of order $\underline{O}(\sqrt{t})$, as t goes to infinity. We will not go into details but note that, bearing in mind the well know diffusion approximation, it is not a surprise.

Assume that $0 < \alpha < \beta < 1$ and introduce *zone-adaptive control strategy* which consists in selection of the starting capital

$$\widehat{u}_{z,t} = \begin{cases} u_{\mathrm{low}}(\alpha, \beta, t), & z < z(\alpha, \beta, t), \\ u_{\mathrm{targ}}(\alpha, t) + z, & z(\alpha, \beta, t) \leqslant z \leqslant 0, \\ u_{\mathrm{targ}}(\alpha, t), & z > 0 \end{cases} \tag{4}$$

and of the premium loading

$$\widehat{\tau}_{z,t} = \begin{cases} \tau_{\max}(\alpha, \beta, t), & z < z(\alpha, \beta, t), \\ -\dfrac{z}{\mathsf{E}V(t)}, & z(\alpha, \beta, t) \leqslant z \leqslant 0, \\ 0, & z > 0, \end{cases} \tag{5}$$

where maximal loading is

$$\tau_{\max}(\alpha, \beta, t) = -\frac{z(\alpha, \beta, t)}{\mathsf{E}V(t)}.$$

Asset–liability and solvency performance. Consider performance of the zone-adaptive strategy in terms of return and solvency.

Targeting. For the strategy (4)–(5) and for $0 < \alpha < \beta < 1$, in both classical and diffusion frameworks

$$\mathsf{E}R_t(\widehat{u}_{z,t}, \widehat{\tau}_{z,t}) = u_{\text{targ}}(\alpha, t) \quad \text{for any } z \in \mathsf{R}.$$

Hitching the annual models together — we will not go into details which are transparent, but involve long nested integrals — this results yields that the capital at the end of every single year equals "in the average" the target value $u_{\text{targ}}(\alpha, t)$.

Solvency. For the strategy (4)–(5) and for $0 < \alpha < \beta < 1$, in both classical and diffusion frameworks

$$\alpha \leqslant \boldsymbol{\psi}_t(\widehat{u}_{z,t}; \widehat{\tau}_{z,t}) = \mathsf{P}\bigl\{ \inf_{0<s\leqslant t} R_s(\widehat{u}_{z,t}, \widehat{\tau}_{z,t}) < 0 \bigr\} \leqslant \beta \quad \text{for any } z \in \mathsf{R}.$$

Hitching the annual models together, this result yields the upper bounds on the probability of ruin of the multiperiod controlled insurance process. In particular, for a time-horizon of n identical insurance years

$$\mathsf{P}\{\text{ruin within } n \text{ years}\} = \sum_{k=1}^{n} \mathsf{P}\{\text{first ruin in year } k\} \leqslant n\beta.$$

Premium rates oscillation range. Evidently, the premium rates oscillation range for a time-horizon of n identical insurance years is bounded from below by zero, and from above by $\tau_{\max}(\alpha, \beta, t)$. In the diffusion framework

$$\tau_{\max}(\alpha, \beta, t) = \frac{\sigma}{\mu\sqrt{t}}\, x_{\alpha,\beta}.$$

Investment power of the zone-adaptive strategy. For the strategy (4)–(5), the random variable

$$S_{z,t} = \begin{cases} 0, & u_{\text{low}}(\alpha,\beta,t) \leqslant R_t(\widehat{u}_{z,t}, \widehat{\tau}_{z,t}) \leqslant u_{\text{targ}}(\alpha, t), \\ R_t(\widehat{u}_{z,t}, \widehat{\tau}_{z,t}) - u_{\text{targ}}(\alpha, t), & R_t(\widehat{u}_{z,t}, \widehat{\tau}_{z,t}) > u_{\text{targ}}(\alpha, t), \\ -(u_{\text{low}}(\alpha,\beta,t) - R_t(\widehat{u}_{z,t}, \widehat{\tau}_{z,t})), & R_t(\widehat{u}_{z,t}, \widehat{\tau}_{z,t}) < u_{\text{low}}(\alpha,\beta,t) \end{cases}$$

is called annual excess (of either sign) of capital.

Dynamic solvency provisions. For the strategy (4)–(5), in both classical and diffusion frameworks

$$\mathsf{E}S_{z,t} > 0 \quad \text{for any } z \in \mathsf{R}.$$

This result shows that application of the zone-adaptive strategy for a time-horizon of n identical insurance years is favorable (in terms of mean values) to dynamic solvency provisions, with the tendency to increase rather than to diffuse.

Some generalizations and discussion. Generalizations of the above framework are numerous.

Non-stationary case. We conjectured that the multiperiod insurance process is composed of the identical annual risk models. The generalization to non-stationary case, when the annual risks remain known but differ throughout the insurance years, is straightforward: endow t, the parameters μ, λ (in the classical risk case), μ, σ (in the diffusion case) and α, β with subscripts indicating the year number, i.e., t_k and μ_k, λ_k, σ_k, α_k, β_k. The arguments yielding the above results remain essentially the same, though behaviour of the insurance process may deteriorate.

Refined control. There may be proposed many other refined control strategies backed by the insurance practice. A straightforward example of the refined control is adaptive selection of the control parameters α, β feed-backed on the past history. Another example addresses solvency control levels (see [**4**]) which are selected to be early warnings of ruin.

Non-Markov modelling. Being a function of the previous-year state of the insurance company only, the control strategy considered above is called Markov. Further generalization consists in non-Markov modelling. Non-Markov control is mentioned, e.g., in Directives [**1**], where three- and seven-years feedback is applied.

Diffusion approximations. The Brownian motion model is of particular interest because of its rôle in diffusion approximation, when the discrete random walk is path-wisely approximated by an appropriate Brownian motion process, for which the probabilities of interest may be computed exactly. This analysis, aiming considerable weakening of the model assumptions, is strategically clear, but requires many technicalities.

Scenario stress testing and unknown risk. Particularly important is generalization on the case of unknown or incompletely known risk. As in the practice, where the company has to devise an information system, i.e., a system for observing the insurance process as it develops, the model has to be supplemented by a function which returns estimated or forecasted values of the risk parameters, to be subsequently used in the control.

Investment and inflation. Commonly, some portions of provisions are invested, and the insurance process is subject to inflation. Investment and inflation may be included into the model, particularly within diffusion framework.

References

[1] *Directive 2002/13/EC of the European Parliament and of the Council of 5 March 2002*, Brussels, 5 March 2002.

[2] *On Solvency, Solvency Assessment and Actuarial Issues.* An IAIS Issues Paper (Final version), IAIS Solvency and Actuarial Issues Subcommittee, 2000.

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