SOME ASPECTS OF RATE MAKING AND COLLECTIVE RISK MODELS WITH VARIABLE SAFETY LOADINGS

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Summary

The problem of rate making and solvency analysis is considered. A modification of the collective risk model that amounts to eventual decreasing of the premium rates, as the initial risk reserve grows, is introduced. Being of greater complexity, this model could better reflect some important aspects of real life, in particular in what concerns competitive insurance markets and comprehensive insurance, and has methodological advances.

The rates of decreasing of the corresponding probabilities of ruin are different from the classical Cramérian exponent. Though only the case of light tailed distributions is considered, a great diversity of the rates including in particular power ones, emerges. The power rates appeared previously only in the context of heavy tailed claim amounts distributions.

QUELQUES ASPECTS DU CALCUL DE TARIF ET LE MODELE DE RISQUE COLLECTIF AVEC LE VARIABLE CHARGEMENT DE SECURITE

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Résumé

Le problème du calcul de tarif et d'analyse de solvabilité est considéré. Une modification du modèle collectif de risque qui consiste en réduction progressive de la mesure des primes d'assurance, avec l'accroissement simultané de réserve de risque initial, est introduite. Quoique étant plus complifié, ce modèle peut tenir compte de quelques aspects importants de la pratique réelle d'assurance, particulièrement concernant des marchés d'assurance compétitif et l'assurance combinée. Outre cela, ce modèle possède quelques avantages métodologiques.

Les vitesses de la réduction des probabilités de ruine correspondantes sont differentes de l'exponentiel Cramérienne classique. Bien que le cas des distributions avec la queue légère est considéré, un large spectre des vitesses qui comprend particulièrement puissances, se déclare. Cette vitesse etait connue autrefois seulement en cas des distributions avec la queue lourde.

1. Rate making, solvency and theory of risk

Insurance is a method of coping with risks, while the object of theory of risk is to give a mathematical analysis of the random fluctuations in an insurance business and to discuss the various means of protection against their inconvenient effects[1].

The basic functions of insurance are underwriting and rating, and the basic standards in rates making are the following. The rates, which are the prices per units of exposure, should reflect fairly the risk involved, they should produce a premium adequate to meet total losses but should not bring unreasonably large profits, they should be revised often enough to reflect the current costs, and finally their structure should tend to encourage loss prevention among those who are insured.

From the viewpoint of the insured person, an insurable risk is one for which the probability of loss is not so high as to require excessive premiums. The insurer, however, needs to assign "loaded" premiums sufficient for business to take its normal course for a long time. For example, in property insurance, only two-thirds of the premium covers the expected cost of loss payment, while approximately one-third covers expenses and profit. These percentage vary somewhat according to the particular type of insurance, but three major rate elements which are

- the loss cost per unit of exposure,
- the administrative expenses, or "loading",
- the profit,

is a rule.

The insurance business is subject to extensive government regulation which is established rather to protect claimants and policyholders than insurers, and rate making and minimum standards for financial solvency are typically among the main aspects of the regulating. Many insurance legislations e.g., in UK, generally accept that in a business so diverse in character it would be useless to try to safeguard solvency by imposing minimum premium rates, and that competition offers the best safeguard to policyholders against the overcharging by the companies.

Most attention from the part of supervision is paid therefore to increasing of premiums rather than to their decreasing, though the crucial importance of potential linkages between the different elements of the insurer's management is recognized. In particular, the US all industry committee is known to seek to provide for as much price competition as possible but at the same time to protect the industry practice of bureau ratemaking because unrestricted competition had resulted in too many insurers insolvencies. In Japan rates are controlled by voluntary rating bureaus under government supervision, and law requires rates to be "reasonable and nondiscriminatory".

As a summary, say that the development of a balanced interplay between different rate elements reflecting

- statutory solvency requirements,
- competition among insurers affecting their internal management strategies,
- market conditions,

[1] See Cramér (1930), p. 7.

is among the main problems of insurance.

The classical theory of risk, as a part of Actuarial Science, focuses its attention on the *outflow* process, looking first at claim numbers, then at a distribution of claim size and finally putting these two together into an aggregate claim amounts process. The *income* process which is the *initial capital* + *premium income* is introduced in a rather simple way[2], growing linearly in time with a constant intensity c. The resulting *surplus process* of an insurance business is generated as *initial capital* + *premium income* − *outflow*.

This bird's eye view of the insurance business, which comply however with its basic principles, has been formalized in the notion of the *collective risk model* of a nonlife insurance company introduced by F. Lundberg in 1903. Interpreted by H. Cramér and developed by E. Sparre Andersen and by a number of researchers, it remains up to now one of the main actuarial models which is concerned with final business results. Paying no attention to individual policies, this model considers the risk business of an insurance company as a whole: claims occur from time to time and have to be settled by the company, while on the other hand the company receives a continuous flow of risk premiums from the policyholders. An important problem within this model is to investigate the *ruin probability*, with the *ruin* interpreted as a technical term indicating insolvency.

Mathematically, the surplus process at any time t is described as the *risk reserve process* starting at time $t = 0$,

$$R(t) = u - \sum_{i=1}^{N(t)} Y_i + ct, \quad t \geqslant 0, \tag{1}$$

where $N(t) = \max\{n \geqslant 1 : \sum_{i=1}^{n} T_i \leqslant t\}$ is the number of claims having occurred up to time t, $u > 0$ is the initial capital, or the initial risk reserve, and $c > 0$ is the gross risk premium intensity. The sequences of the random variables (r.v.) $\{T_i\}_{i\geqslant 1}$, being intervals between claims, and $\{Y_i\}_{i\geqslant 1}$, being the amounts of claims, satisfy usually a number of simplifying assumptions e.g., are independent and identically distributed, or comply with some other distributional requirements.

Ruin occurs at time s if $R(s) < 0$, and the probability that ruin occurs within the time interval $(0, t]$ is

$$\psi(t, u) = \mathbf{P}\{\inf_{0<s\leqslant t} R(s) < 0\}. \tag{2}$$

The probability of ultimate ruin is $\psi(u) = \psi(+\infty, u)$.

The amount $\tau = c\mathbf{E}T_1/\mathbf{E}Y_1 - 1$ called *relative safety loading* of the insurance business is an important characteristic of the risk reserve process. Indeed, since $c\,T_i$ is the premium acquired and Y_i is the claims amount paid out on the i-th "step" which is time between $(i-1)$-th and i-th claims, the condition $\tau > 0$ means that successful "steps" are persistent. Otherwise, $\tau < 0$ means that successful "steps" are rare and, in total, the business is ruinous. Thus, being a positive constant, safety

[2] Though derivatives such as investment, borrowing and inflation could be considered on more advanced stages.

loading makes both probabilities of ultimate ruin $\psi(u)$ and of the ruin within finite time $\psi(t, u)$ small, as u is sufficiently large.

Common sense is required in deciding whether and to what extent the proposed model is competent to describe the phenomena of concern[3]. Even though the collective risk model is typically inadequate to describe real world phenomena, it is an element in the more advanced modeling. It could incorporate many factors such as volatility of the assets and the impact of inflation, which normally affects both investment behavior and claim settlement, and be concentrated on such problems as reserve estimation, premium rating and profitability, expenses, reinsurance, catastrophic claims and interaction with the rest of the insurance market. The last problem will be among our main concerns.

2. Collective risk model and some aspects of insurance business

Some principal limitations of the collective risk model, as it is described above, consist both in its basic structure assumptions and in certain technical requirements on the outflow process, though just the later have attracted most attention. One of the limitations consists in the fact that we have avoided assuming any dependence between the initial capital u and the premium rates c, though a typical further assumption is that the initial capital grows. Evidently, the assumptions on growing initial capital and on premium income growing linearly in time while the business runs, accentuate the income process, though the outflow remains independent on both these assumptions.

Besides some lack of balance in the model, it does not reflect several important business aspects, a principal of them being that insurer should not be considered in isolation, without explicit regard for the interplay between other insurers. While an insurer operating in a competitive insurance market is implicated, a suggestion prompts that policy prices are among the primary influences of this interplay. We start with a few reminiscences of the insurance practice to illustrate this idea.

Most insurance managers closely follow the underwriting cycles and increase rates if the risk business is disquietingly bad. Conversely, if a company with the initial value u of the risk reserve is managed in such a way that the probability of ruin at future moments has a value regarded as sufficiently small, after some time it is very likely that the risk reserve will reach a value considerably larger than the initial value u. When this situation has arrived, it would obviously be plausible to decrease rates without increasing the probability of future ruin. This would mean that either premiums themselves will be decreased, or a larger part of the premiums actually paid by the policyholders would be at the free disposal of the company, for bonus or otherwise.

Intuition suggests that a larger company could reduce deliberately its policy prices, in contrast with a smaller one, for the initial capital seems to be here one of

[3] Daykin at al. (1996), p. xiii, wrote: "There is often perceived to be a wide gap between practical actuaries and the protagonists of risk theory. This has been exacerbated by the very theoretical nature of many presentations of risk theory. However, it is the authors' belief that practical actuaries must be competent in the analysis of uncertainty, as evidenced by the British Institute of Actuaries motto *certum ex incertis* (certainty out of uncertainty)."

the most important factors. Depending on context it could be interpreted as fair or attractive (customers pay less money) or competitive or even dumping (larger insurance company dictates its prices) marketing politics. Once this reduction is considered desirable, the determination of "safe" limits within which such reduction could be performed, becomes of primary interest for insurer.

Furthermore, one may consider it plausible to assume premium rates decreasing as initial capital grows to achieve an improvement matching between premium income and associated expenditures. For example, performance in collecting premiums could decrease as a part of a comprehensive risk management process. If to equalize profits and losses between tariff classes with different outcomes is permissible, a company has actual premium income as a sum of incomes rendered by premiums in each class, sometimes showing profits and sometimes losses. Once operating on a competitive market, a company could prefer not to refuse to deal with certain classes even if a profit rendered in them separately could be fluctuating or even negative. Say again that intuition suggests that a small company is not able to do so, in contrast with a larger one.

Finally, a company which capital is growing, though performance in collecting premiums decreases due to some grave management errors, suggests the above mentioned adjustment in the risk model. An example of such an error could be lack of balance between premium income and associated expenditures, inadmissible large and permanent expenses (expenses of administration, long-term investments in non-liquid assets, and so on) loaded into the premiums, etc.[4]

An attempt to translate these considerations into mathematical statements in the framework of the collective risk model leads to the following: the safety loading, as a part of premiums, should depend on the initial capital. Introduce the following definitions.

Definition 1. We say that the risk premium intensity c_u dependent on u is *asymptotically reduced of order* τ_u, if

$$c_u = (1 + \tau_u)\mathbf{E}Y_1/\mathbf{E}T_1, \tag{3}$$

with $\tau_u > 0$, $\tau_u \to 0$, as $u \to \infty$.

Evidently, this equality is the same as $\tau_u = c_u\mathbf{E}T_1/\mathbf{E}Y_1 - 1$, and the equivalent assumption is that the relative safety loading is positive but tending to zero, as $u \to \infty$.

Definition 2. We say that the risk premium intensity c_u dependent on u is *asymptotically admissible* if $\psi(u) \to 0$, as $u \to \infty$. If $\psi(u) \to 1$, as $u \to \infty$, it is called *asymptotically deficient*[5].

Asymptotical deficiency means that the probability of ultimate ruin will be tending to 1 indispensably, and the insurance business will be asymptotically ruinous. The proof of the fact of admissibility is an important problem. Moreover, to determine the rates of premiums payable which make the probability $\psi(u)$ to be of

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[4] An extra hint that a growing capital might lead to decreasing of business activity and growing of indirect expenditure is the classical Law of Diminishing Returns by von Neumann and Morgenstern.

a required magnitude, is particularly important.

Premiums dependent on the initial capital seems a reasonable, though merely a first step in our attempt to introduce finer market effects in the collective risk model. In some cases it might seem more realistic to assume that premium rates depend on the *current value* of the risk reserve[6] rather than on the initial capital. However, to try to approve our assumption note that e.g., in the automobile insurance, rates are revised annually or even more often, but nevertheless they tend to be out of date.

From the other side, it is quite clear that a complete and strict simultaneous treatment of several insurers, taking decisions independently, which is an essence of the competitive insurance market, adds considerably to the complexity of the problem, making it largely intractable, even with the use of simulation technique.

We complete this Section with a citation from Daykin et al. that "stochastic studies of insurance markets are still at a very preliminary stage and awaiting substantial further research efforts. Market effects are, however, too important to leave out and it is worthwhile adopting even a very approximate approaches to explore the possible impact"[7].

3. Probabilities of ultimate ruin in the "classical" model

Assuming that Y_i and T_i, $i = 1, 2, \dots,$ are mutually independent sequences of independent and exponential r.v. with parameters $\mu > 0$ and $\lambda > 0$ respectively, the relative safety loading transforms into $\tau = c\mu/\lambda - 1$ and is positive if $c > \lambda/\mu$. These assumptions define a "classical" risk model which is a particular case of the Cramérian risk model[8] known to be sufficiently realistic to serve as a first approximation that may be practically used in many cases.

The "classical" model is mathematically tractable. It is an exceptional model, for which an *exact* formula for the probability of ultimate ruin, going back to Lundberg (1926) and Cramér (1930),

$$\psi(u) = \frac{1}{1+\tau} \exp\left\{-\frac{\mu\tau}{1+\tau}u\right\}, \tag{4}$$

true for *any* $u > 0$, is evaluated[9].

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[5] Note that in the framework of variable premium intensities the intermediate cases are possible (see Example 4 below).

[6] See e.g., Cramér (1955), p. 85: "It might seem natural to introduce a priori a *variable safety loading*, laying down the rule that the safety loading received by the risk reserve would amount to $\tau(x)dt$ during a time element dt when the value of the risk reserve is x. Here $\tau(x)$ would be an a priori given function of x, and it seems natural to require that $\tau(x)$ should be never increasing as x increases."

[7] Daykin et al. (1996), p. 373.

[8] Once relaxing the assumptions on the distribution of Y_i and on the mutual independence of Y_i and T_i, one gets the Cramérian model. The assumption on exponential T_i is however essential since entails the Poissonian claims arrival.

[9] See e.g., (12.3.8) in Bowers et al. (1986).

Evidently, the almost trivial assertion[10] that either $\tau \to 0$, or $\tau < 0$, implies $\psi(u) = 1$, that is, "certainty of ruin", gets rather different if $\tau_u \to 0$, as a function of $u \to \infty$.

Example 1. Assume that in the classical risk model $\lambda = \mu = 1$. First, let the safety loading be constant, $\tau = 0.2$. Then $c = 1.2$ and

$$\psi_c(u) = \frac{5}{6} \exp\left\{-\frac{u}{6}\right\}. \tag{5}$$

Second, let the safety loading be dependent on u and vanishing as u grows to infinity. For example, we impose the following rates on the convergence: $\tau_u = (\ln u)^{-2}$. Evidently,

$$\psi_v(u) = \frac{(\ln u)^2}{1 + (\ln u)^2} \exp\left\{-\frac{u}{1 + (\ln u)^2}\right\}. \tag{6}$$

Here and in what follows, subscripts *"c"* and *"v"* will refer to the models with a constant and a variable safety loading, respectively.

Evidently, $\psi_c(u^*) = \psi_v(u^*)$ for $u^* = \exp(\sqrt{5}) \approx 9.35$. The values of $\psi_c(u)$ and $\psi_v(u)$ for $u > u^*$ found numerically are shown in table 1. The discrepancies are seen to grow significantly, as u is growing, but $\psi_v(u)$ is in no case tending to 1, as $u \to \infty$.

An evident conclusion is that even if we have the same outflow process and if we are starting with the same initial capital $u = u^*$ which equalizes the starting values of the probabilities of ruin, differences in premium structures lead eventually to a quite different ruin probabilities dynamics, as u grows.

Table 1. *Probabilities of ultimate ruin $\psi_c(u)$ and $\psi_v(u)$.*

u	$\psi_c(u)$	$\psi_v(u)$
9.35	0.1752	0.1752
10.35	0.1484	0.1703
11.35	0.1256	0.1650
12.35	0.1063	0.1597
13.35	0.0900	0.1542
20	0.0297	0.1211
30	0.0056	0.0845
40	0.0001	0.0602

4. Approximations of the probabilities of ruin

Andersen's, or renewal, risk model is a generalization of the Cramérian model designed to bring into consideration the possibility of contagion between claims[11].

[10] See e.g., Bowers et al. (1986), p. 350, 12 line below.

[11] The type of contagion, which may be considered, is characterized by the property that a claim is more likely (or, if that should be wanted, less likely) to occur shortly after another claim, and that the probability of occurence of claims depends on the time elapsed since the last claim and only on this quantity (Andersen (1957), p. 219).

Renewal claims arrival processes do not look as a mere analytical overcomplication though since Andersen's model has been introduced in 1957, a number of authors claimed that "no practical examples of a renewal risk process other than Poissonian have been produced"[12]. Indeed, e.g., modern mass media and telecommunication networks could introduce substantial and sometimes unpredictable dependence into behavior of policyholders and the a priori assumption on the Poissonian origin of claims arrival might be suspicious.

An other attempt to go apart from a merely Poissonian claims arrival are Cox processes which describe insurance schemes where the whole risk situation varies with variations in the environment. Though the initial motivations were different, Kingman's (1964) criterion revealed that Cox and Andersen's renewal claims arrivals are identical mathematical objects in special cases.

4.1 Constant safety loading

Put for simplicity $X_i = Y_i - cT_i$ and assume that Y_i and T_i have bounded probability density functions w.r.t. Lebesgue measure. The famous asymptotic formulas for the probabilities of ruin in Andersen's risk model involve *adjustment coefficient* $\varkappa$[13] which is a positive solution of the *Lundberg equation* $\mathbf{E}\exp(\varkappa X_1) = 1$, the basic results being the *Lundberg inequality*

$$\psi_c(u) \leqslant e^{-\varkappa u}, \tag{7}$$

the *Cramér – Lundberg approximation*

$$\lim_{u\to\infty} e^{\varkappa u}\psi_c(u) = C, \tag{8}$$

where C is the *Cramér – Lundberg constant*, and the approximation

$$\lim_{u\to\infty}\sup_{t\geqslant 0}|\psi_c(t,u)e^{\varkappa u} - C\,\Phi_{(mu,D^2u)}(t)| = 0, \tag{9}$$

where $\Phi_{(mu,D^2u)}(t)$ stands for the Normal probability distribution function with mean mu and variance D^2u. The general expressions for m, D^2 and C are known (see e.g., (7.2) in von Bahr (1974)), though except in the "classical" particular case[14], their calculation is known to be rather difficult.

The Normal approximation (9) was obtained first by von Bahr (1974) and has been refined in Malinovskii (1994, 1996). Other, e.g., diffusion, approximations for $\psi_c(t,u)$ have been developed (see e.g., Asmussen (1984)).

4.2 Variable safety loading

Put $X_{u,i} = Y_i - c_uT_i$, $i = 1, 2, \ldots$. For $S_{n,u} = \sum_{k=1}^n X_{u,k}$, $n = 1, 2, \ldots$, for the probability distribution function $B_u(x,y) = \mathbf{P}\{X_{u,1} \leqslant x, T_1 \leqslant y\}$ and for a positive solution $\varkappa_u$ of the Lundberg equation

$$\mathbf{E}\exp(\varkappa_u X_{u,1}) = 1 \tag{10}$$

[12] See e.g., Seal (1974), p. 121.

[13] See e.g., Bowers et al. (1986), p. 351; having in mind the Cramér – Lundberg approximation, $\varkappa$ is also termed the *Lundberg exponent*.

[14] Where $m = \mu/(\lambda\tau(1+\tau))$, $D^2 = 2\mu/(\lambda^2\tau^3)$, and $C = 1/(1+\tau)$.

define

$$\overline{\beta}_u(t_1,t_2) = \iint e^{i(t_1x+t_2y)}\overline{B}_u(dx,dy),$$

$$\rho_u(t_1,t_2) = \sum_{n=1}^{\infty}\frac{1}{n}\iint_{x\leqslant 0} e^{i(t_1x+t_2y)}\overline{B}_u^{*n}(dx,dy),$$

$$\nu_u^{i,j} = \mathbf{E}X_{u,1}^i T_1^j,\ \overline{\nu}_u^{i,j} = \mathbf{E}X_{u,1}^i T_1^j \exp(\varkappa_u X_{u,1}),\ i,j=0,1\ldots,$$

where $\overline{B}_u(dx,dy) = e^{\varkappa_u x}B_u(dx,dy)$ and the asterisk denotes convolution. The following basic result is the Theorem 2.1 proved in Malinovskii (1997).

Theorem 1. *In the risk model with c_u asymptotically reduced of order τ_u assume that*

(1) $|\overline{\beta}_u(t_1,t_2)|^p$ *and* $|\rho_u(t_1,t_2)|^p$ *are integrable for some* $p\geqslant 1$ *and* u *sufficiently large,*

(2) *for a right neighborhood of zero* $\mathbb{N}$ *and for a constant* $H>0$

$$\sup_{u\in\mathbb{N}}\mathbf{E}\exp(hX_{u,1})<\infty,\ \sup_{u\in\mathbb{N}}\mathbf{E}\exp((h+\varkappa_u)X_{u,1})<\infty \text{ for } |h|<H, \tag{11}$$

(3) $D^2 = \lim_{u\to 0} D_u^2 > 0$.

Then

$$\sup_{t\geqslant 0}\left|\psi_v(t,u) - C_u e^{-\varkappa_u u}\Phi_{(m_u u, D_u^2 u)}(t)\right| = \underline{O}\big((\tau_u u)^{-1/2}e^{-\varkappa_u u}\big), \tag{12}$$

as $u\to\infty$*, where*

$$m_u = \overline{\nu}_u^{0,1}/\overline{\nu}_u^{1,0},\quad D_u^2 = ((\overline{\nu}_u^{0,1})^2\overline{\nu}_u^{2,0} - 2\overline{\nu}_u^{1,0}\overline{\nu}_u^{0,1}\overline{\nu}_u^{1,1} + (\overline{\nu}_u^{1,0})^2\overline{\nu}_u^{0,2})/(\overline{\nu}_u^{1,0})^3,$$
$$C_u = \frac{1}{\varkappa_u\overline{\nu}_u^{1,0}}\exp\Big(-\sum_{n=1}^{\infty}\frac{1}{n}\big[\mathbf{P}(S_{n,u}>0) + \mathbf{E}e^{\varkappa_u S_{n,u}}\mathbf{1}_{(S_{n,u}\leqslant 0)}\big]\Big). \tag{13}$$

The proof of (12) is far from being a straightforward extension of (9). Mention, in passing, that the renewal approach used in von Bahr (1974) to prove (9) encounters substantial technical difficulties since the *series* $\{X_{u,i}\}_{i\geqslant 1}$ of independent r.v. in the framework of variable safety loading took place of the *sequences* $\{X_i\}_{i\geqslant 1}$ in the traditional scheme.

Formulating (14) – (16), we will be based on the Theorems 2.2, 3.1, 3.2, 3.3 and 4.1 proved in Malinovskii (1997). Put $\tau_u' = \tau_u\mathbf{E}Y_1$. While in the "classical" case $\varkappa_u$, m_u, D_u^2, C_u were found in a closed form[15]

$$\varkappa_u = \frac{\mu\tau_u}{1+\tau_u},\quad m_u = \frac{\mu}{\lambda\tau_u(1+\tau_u)},\quad D_u^2 = \frac{2\mu}{\lambda^2\tau_u^3},\quad C_u = \frac{1}{1+\tau_u} \tag{14}$$

(see Theorem 2.2 in Malinovskii (1997)), in the general Andersen's case one has merely $C_u = 1+\overline{o}(1)$,

$$\begin{aligned}
\varkappa_u &= \mathfrak{a}_1\tau_u' + \mathfrak{a}_2{\tau_u'}^2 + \cdots + \mathfrak{a}_N{\tau_u'}^N + \cdots,\\
m_u &= \mathfrak{m}_{-1}{\tau_u'}^{-1} + \mathfrak{m}_0 + \cdots + \mathfrak{m}_{N-1}{\tau_u'}^{N-1} + \cdots,\\
D_u^2 &= \mathfrak{v}_{-3}{\tau_u'}^{-3} + \mathfrak{v}_{-2}{\tau_u'}^{-2} + \cdots + \mathfrak{v}_{N-2}{\tau_u'}^{N-2} + \cdots,
\end{aligned} \tag{15}$$

[15] Quite suggestive; however, the proof of validity of (12) will remain a real problem.

with the series convergent for all sufficiently large u in the conditions of the Theorem 1.

The proof of (15) applies as a main tool the Bürmann–Lagrange theorem. A technique for calculation of the explicit expressions for $\mathfrak{a}_i$, $\mathfrak{m}_i$, $\mathfrak{v}_i$ with i arbitrary, easy to algorithmize and to set up as a computer routine, has been developed. In particular, it yields

$$\begin{aligned}
&\mathfrak{a}_1 = 2\,\frac{\gamma_{01}^2}{\gamma_{20}}, \quad \mathfrak{a}_2 = 4\,\frac{\gamma_{01}^2}{\gamma_{20}^2}\Big(\gamma_{11} - \frac{1}{3}\frac{\gamma_{01}\gamma_{30}}{\gamma_{20}}\Big),\\
&\mathfrak{a}_3 = -2\,\frac{\gamma_{01}^2}{\gamma_{20}^2}\Big(\gamma_{02} - 4\frac{\gamma_{11}^2}{\gamma_{20}}\Big) + 4\,\frac{\gamma_{01}^3}{\gamma_{20}^3}\Big(\gamma_{21} - 2\frac{\gamma_{30}\gamma_{11}}{\gamma_{20}}\Big) - \frac{2}{3}\,\frac{\gamma_{01}^4}{\gamma_{20}^4}\Big(\gamma_{40} - \frac{8}{3}\,\frac{\gamma_{30}^2}{\gamma_{20}}\Big),\\
&\mathfrak{m}_{-1} = \gamma_{01},\ \mathfrak{m}_0 = \frac{2\gamma_{01}\gamma_{11}}{\gamma_{20}} - \frac{\gamma_{30}\gamma_{01}^2}{3\gamma_{20}^2}, \quad \mathfrak{m}_1 = -\frac{2\gamma_{01}\gamma_{02}}{\gamma_{20}} + \frac{\gamma_{01}}{\gamma_{20}^2}(4\gamma_{11}^2 \qquad (16)\\
&+ 3\gamma_{01}\gamma_{21}) - \frac{\gamma_{01}^2}{3\gamma_{20}^3}(\gamma_{01}\gamma_{40} + 10\gamma_{11}\gamma_{30}) + \frac{5\gamma_{01}^3\gamma_{30}^2}{9\gamma_{20}^4}, \quad \mathfrak{v}_{-3} = \gamma_{20}, \quad \mathfrak{v}_{-2} = 0,
\end{aligned}$$

where $\gamma_{ij} = \mathbf{E}(Y_1\mathbf{E}T_1 - T_1\mathbf{E}Y_1)^i T_1^j$, $i, j = 0, 1 \ldots$.

4.3 Three illustrative examples

Example 2. Consider two risk models of the Example 1, the first one corresponding to the constant safety loading $\tau = 0.2$, while in the second $\tau_u = (\ln u)^{-2}$ is dependent on u. The equality (14) and the approximations (9), (12) yield for u sufficiently large

$$\psi_c(t,u) \approx \frac{5}{6}\exp\Big\{-\frac{u}{6}\Big\}\,\Phi_{(25u/6,\ 250u)}(t), \qquad (17)$$

and

$$\psi_v(t,u) \approx \frac{(\ln u)^2}{1+(\ln u)^2}\exp\Big\{-\frac{u}{1+(\ln u)^2}\Big\}\,\Phi_{(u(\ln u)^4/(1+(\ln u)^2, 2u(\ln u)^6)}(t). \qquad (18)$$

Though the right hand sides of (17) and (18) coincide for $u = u^* \approx 9.35$, the discrepancy is visible for $u = 11.35$ (see table 2), gets substantial for $u = 13.35$ (see table 3), and grows dramatic for $u = 20$ (see table 4).

TABLE 2. *Approximations for the probabilities of ruin $\psi_c(t, 11.35)$ and $\psi_v(t, 11.35)$.*

t	$\psi_c(t, 11.35)$	$\psi_v(t, 11.35)$
25	0.0424	0.0525
50	0.0653	0.0755
75	0.0877	0.0994
100	0.1054	0.1212
125	0.1166	0.1385
150	0.1223	0.1506
175	0.1246	0.1581
200	0.1254	0.1620

Table 3. *Approximations for the probabilities of ruin $\psi_c(t, 13.35)$ and $\psi_v(t, 13.35)$.*

t	$\psi_c(t, 13.35)$	$\psi_v(t, 13.35)$
50	0.0415	0.0582
100	0.0701	0.0920
150	0.0854	0.1216
200	0.0894	0.1407
250	0.0900	0.1499
300	0.0900	0.1532

Table 4. *Approximations for the probabilities of ruin $\psi_c(t, 20)$ and $\psi_v(t, 20)$.*

t	$\psi_c(t, 20)$	$\psi_v(t, 20)$
50	0.0094	0.0310
100	0.0176	0.0434
150	0.0245	0.0573
200	0.0282	0.0714
250	0.0294	0.0846
300	0.0296	0.0959
350	0.0297	0.1049
400	0.0297	0.1114

Example 3. The calculation of $\varkappa_u$, m_u, D_u^2, C_u seems particularly simple in the "classical" case. In the general Andersen's model one ought to apply (15). We turn now to calculation of the coefficients $\mathfrak{a}_i$, $\mathfrak{m}_i$, $\mathfrak{v}_i$ in a model different from the "classical".

To have an illustration, consider the most suggestive generalization of the Cramérian model often used in applications and in theoretical contexts: assume that (iid) amounts of claims $\{Y_i\}_{i\geqslant 1}$ are Gamma with shape parameter $\beta > 0$ and scale parameter $\mu > 0$ and that (iid) inter-occurence times $\{T_i\}_{i\geqslant 1}$ are Gamma with shape parameter $\alpha > 0$ and scale parameter $\lambda > 0$. Moreover, these sequences are assumed mutually independent.

Recall that the Gamma densities are

$$f_T(x) = \begin{cases} \lambda x^{\alpha-1} e^{-\lambda x}/\Gamma(\alpha), & x > 0, \\ 0, & x \leqslant 0, \end{cases} \quad f_Y(t) = \begin{cases} \mu x^{\beta-1} e^{-\mu t}/\Gamma(\beta), & t > 0, \\ 0, & t \leqslant 0, \end{cases} \tag{19}$$

with $\mathbf{E}T_1^k = \alpha(\alpha+1)\ldots(\alpha+k-1)\lambda^{-k}$, $\mathbf{E}Y_1^k = \beta(\beta+1)\ldots(\beta+k-1)\mu^{-k}$, $k = 1, 2, \ldots$, so that $\mathbf{E}T_1 = \alpha/\lambda$, $\mathbf{E}Y_1 = \beta/\mu$, and $\gamma_{ij} = \mathbf{E}(Y_1\mathbf{E}T_1 - T_1\mathbf{E}Y_1)^i T_1^j$, $i, j = 0, 1 \ldots$.

One easily has

$$\gamma_{ij} = \frac{\alpha^i i\,!}{\lambda^i} \sum_{m=0}^{i} \frac{(-1)^m}{m\,!(i-m)\,!} \frac{\lambda^m \beta^m}{\alpha^m \mu^m} \mathbf{E}T_1^{m+j} \mathbf{E}Y_1^{i-m}, \tag{20}$$

which yields in particular

$$\gamma_{01} = \frac{\alpha}{\lambda},\ \gamma_{20} = \frac{\alpha\beta(\alpha+\beta)}{\lambda^2\mu^2},\ \gamma_{11} = -\frac{\alpha\beta}{\lambda^2\mu},\ \gamma_{02} = \frac{\alpha(\alpha+1)}{\lambda^2},$$
$$\gamma_{30} = \frac{2\alpha\beta(\alpha^2-\beta^2)}{\lambda^3\mu^3},\ \gamma_{21} = \frac{\alpha\beta(\alpha^2+\alpha\beta+2\beta)}{\lambda^3\mu^2},\ \gamma_{12} = -\frac{2\alpha\beta(\alpha+1)}{\lambda^3\mu},$$
$$\gamma_{40} = \frac{3\alpha\beta(\alpha+\beta)[(\alpha+\beta)\alpha\beta+2(\alpha^2-\alpha\beta+\beta^2)]}{\lambda^4\mu^4},$$

and

$$\mathfrak{a}_1 = \frac{2\alpha\mu^2}{\beta(\alpha+\beta)},\ \mathfrak{a}_2 = -\frac{4\alpha\mu^3(2\alpha+\beta)}{3\beta^2(\alpha+\beta)^2},\ \mathfrak{a}_3 = \frac{2\alpha\mu^4(14\alpha^2+17\alpha\beta+5\beta^2)}{9\beta^3(\alpha+\beta)^3}. \tag{21}$$

In the particular case of exponential T_1 which corresponds to $\alpha = 1$, and exponential Y_1, which corresponds to $\beta = 1$, the expressions (21) reduce to $\mathfrak{a}_1 = \mu^2$, $\mathfrak{a}_2 = -\mu^3$, $\mathfrak{a}_3 = \mu^4$, which agrees with the Taylor expansion of the exact expression for $\varkappa_u$ given in (14). Furthermore,

$$\mathfrak{m}_{-1} = \frac{\alpha}{\lambda}, \quad \mathfrak{m}_0 = -\frac{2\mu\alpha(\alpha+2\beta)}{3\lambda\beta(\alpha+\beta)}, \quad \mathfrak{m}_1 = \frac{2\mu^2\alpha(\alpha^2+10\alpha\beta+7\beta^2)}{9\lambda\beta^2(\alpha+\beta)^2},$$
$$\mathfrak{v}_{-3} = \frac{\alpha\beta(\alpha+\beta)}{\lambda^2\mu^2}, \quad \mathfrak{v}_{-2} = 0. \tag{22}$$

In the particular case of exponential T_1 which corresponds to $\alpha = 1$, and exponential Y_1, which corresponds to $\beta = 1$, the expressions (22) reduce to $\mathfrak{m}_{-1} = 1/\lambda$, $\mathfrak{m}_0 = -\mu/\lambda$, $\mathfrak{m}_1 = \mu^2/\lambda$, $\mathfrak{v}_{-3} = 2/(\lambda\mu)^2$, $\mathfrak{v}_{-2} = 0$, which agrees with the Taylor expansions of the exact expressions for m_u and D_u^2 given in (14).

Though this example is intentionally simple, recall that the renewal process $N(t)$ with Gamma interclaim times is a Cox process if $0 < \alpha \leqslant 1$.

Example 4. This example illustrates a great diversity of the new rates of decreasing of the probabilities of ruin. For simplicity, switch again to the "classical" risk model and impose different rates of convergence of τ_u to zero, as $u \to \infty$.

First, let $c_u = \lambda\mu^{-1}(1 + u^{-1})$, or let c_u be asymptotically reduced of order u^{-1}. The equality (4) yields

$$\psi_v(u) \approx \exp\{-\mu\}, \tag{23}$$

which shows that the probability of ruin $\psi_v(u)$ may be tending to an arbitrary real number in the range $[0, 1]$. It is never the case of $\psi_c(u)$ since the limits 0 or 1 constitute the unique alternative.

Second, let $c_u = \lambda\mu^{-1}(1 + u^{-1}\ln u)$, or let c_u be asymptotically reduced of order $u^{-1}\ln u$. The equality (14) and the approximation (12) yield for u sufficiently large

$$\psi_v(u) \approx u^{-\mu},$$
$$\psi_v(t, u) \approx u^{-\mu}\Phi_{(\mu u^2/(\lambda \ln u),\ 2\mu u^4/(\lambda^2(\ln u)^3))}(t), \tag{24}$$

which shows that even though the light tailed distributions are considered, the probabilities of ruin $\psi_v(u)$ and $\psi_v(t, u)$ might be decreasing as a power of u. It is known that for $\psi_c(u)$ and $\psi_c(t,u)$ such rates have emerged only in the case of Y_1 with a heavy tailed distribution.

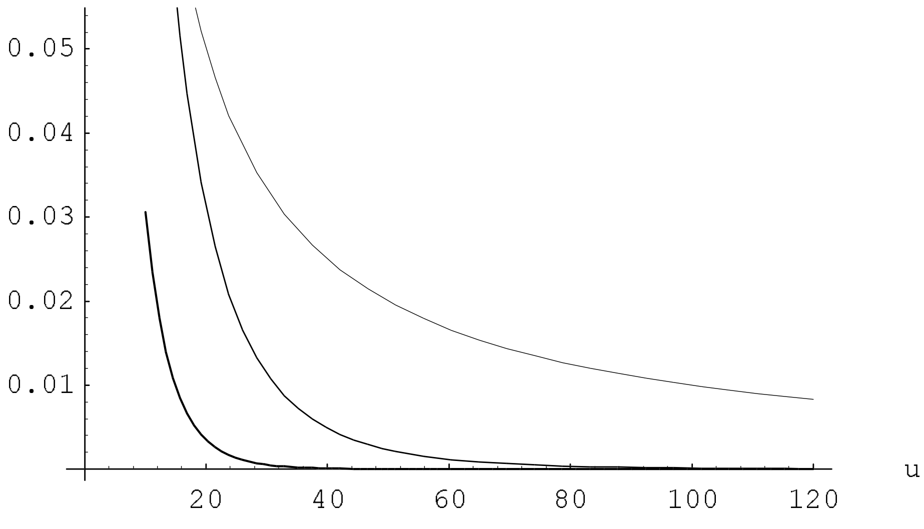
Third, let $c_u = \lambda\mu^{-1}(1+u^{-1/2})$, or let c_u be asymptotically reduced of order $u^{-1/2}$. The equality (14) yields $\varkappa_u = \mu u^{-1/2}(1+u^{-1/2})^{-1} \approx \mu(u^{-1/2} - u^{-1} + u^{-3/2} + \dots)$, $m_u = \mu u^{1/2}\lambda^{-1}(1+u^{-1/2})^{-1} \approx \mu\lambda^{-1}(u^{1/2} - 1 + u^{-1/2} - u^{-1} + \dots)$, $D_u^2 = 2\mu u^{3/2}\lambda^{-2}$, and the approximation (12) yields for u sufficiently large

$$\begin{aligned} \psi_v(u) &\approx \exp\left\{-\mu(u^{1/2}-1)\right\}, \\ \psi_v(t,u) &\approx \exp\left\{-\mu(u^{1/2}-1)\right\}\Phi_{(\mu(u^{3/2}-u+\sqrt{u}-1)/\lambda,\, 2\mu u^{5/2}/\lambda^2)}(t). \end{aligned} \tag{25}$$

Finally, let $c_u = \lambda\mu^{-1}(1+u^{-1/3})$, or let c_u be asymptotically reduced of order $u^{-1/3}$. The equality (14) yields $\varkappa_u = \mu u^{-1/3}(1+u^{-1/3})^{-1} \approx \mu(u^{-1/3} - u^{-2/3} + u^{-1} + \dots)$, $m_u = \mu u^{1/3}\lambda^{-1}(1+u^{-1/3})^{-1} \approx \mu\lambda^{-1}(u^{1/3} - 1 + u^{-1/3} - u^{-2/3} + u^{-1} + \dots)$, $D_u^2 = 2\mu u\lambda^{-2}$, and the approximation (12) yields for u sufficiently large

$$\begin{aligned} \psi_v(u) &\approx \exp\left\{-\mu(u^{2/3}-u^{1/3}+1)\right\}, \\ \psi_v(t,u) &\approx \exp\left\{-\mu(u^{2/3}-u^{1/3}+1)\right\}\Phi_{(\mu(u^{4/3}-u+u^{2/3}-u^{1/3}+1)/\lambda,\, 2\mu u^2/\lambda^2)}(t). \end{aligned} \tag{26}$$

Figure 1. *Approximations for the probabilities of ruin $\psi_v(u)$ for $\tau_u = u^{-1}\ln u$ (thin line), $\tau_u = u^{-1/2}$ (medium line), $\tau_u = u^{-1/3}$ (thick line).*



Remark. In fact, which is particularly transparent in (25) and (26), the truncated expansions

$$\begin{aligned} \varkappa_u u &= \mathfrak{a}_1 u\tau_u' + \mathfrak{a}_2 u{\tau_u'}^2 + \dots + \mathfrak{a}_M u{\tau_u'}^M + \overline{o}(1), \quad u \to \infty, \\ m_u u &= \mathfrak{m}_{-1} u{\tau_u'}^{-1} + \mathfrak{m}_0 u + \mathfrak{m}_1 u\tau_u' + \dots + \mathfrak{m}_M u{\tau_u'}^M + \overline{o}(1), \quad u \to \infty, \\ D_u^2 u &= \mathfrak{v}_{-3} u{\tau_u'}^{-3} + \mathfrak{v}_{-2} u{\tau_u'}^{-2} + \mathfrak{v}_{-1} u{\tau_u'}^{-1} + \dots + \mathfrak{v}_M u{\tau_u'}^M + \overline{o}(1), \quad u \to \infty, \end{aligned}$$

where $M = \inf\{k \geqslant 1 : \lim_{u\to\infty} \tau_u^k u = 0\} - 1$, are often required to use in (12) instead of the full expansions (15) and so much the more of (14). This M could be either infinite if e.g., $\tau_u = 1/\ln u$, or finite if e.g., τ_u decreases as any power $(1/u)^k$, as $0 < k < 1$.

We bound ourselves by these examples, though many other illustrations of the general results could be easily supplied.

5. Some trends of further development

The research within the framework of variable premium intensities has many trends of further development, both innovative and applied.

To mention one among the later, note that to make a practical implementation of the approximations of the probabilities of ruin (12) – (16), one has to derive the coefficients $\mathfrak{a}_i$, $\mathfrak{m}_i$, $\mathfrak{v}_i$ explicitly, like in the Example 3 above. It is shown to be a many stages procedure. First, one has to find out the general expressions like in (16) and then to calculate the coefficients in a final form. Though these calculations require rather cumbersome algebra, they could be algorithmized (see Section 3 in Malinovskii (1997)). To set up these calculations as a computer routine would be desirable.

We mention a few other directions which might look interesting.

5.1 Heavy tailed interclaim distributions

The author's conjecture is that the uniform Cramér conditions (11) in Theorem 1 could be replaced by their "unilateral" counterparts which will make it possible to consider interclaim times with heavy-tailed distributions.

It would be of a particular importance to compare the approximations (12) – (16) with the exact values of the probabilities of ruin. An approach to the calculation of the exact values in case of heavy-tailed T_1 and exponential Y_1 has been proposed in Malinovskii (1997 a). These results develop the numerical technique used previously in case of Poissonian claims arrival processes (see e.g., Seal (1974)).

5.2 Corrected approximations

Numerical comparison of the normal approximation (9) with the exact values of $\psi_c(t,u)$ in the "classical" case performed in Section 4 of Asmussen (1984) shows that the fit of the approximation improves as u increases, but is rather poor even for quite large u: it turns out that the dependence on higher cumulants is not negligible. To produce a correction of (9), Asmussen (1984) suggested an approximation based on a certain heuristic assumption of independence which is however correct only in the case of the "classical" model (see (4.7) in Asmussen (1984)).

In the general Andersen's model and under the assumptions similar to those which have been imposed by von Bahr (1974), an Edgeworth-like correction of (9) has been obtained in Malinovskii (1994):

$$\sup_{t\geqslant 0} \left|\psi_c(t,u)e^{\varkappa u} - C\left(\Phi_{(mu,D^2u)}(t) - Q_1(t(u))\varphi_{(mu,D^2u)}(t)\right)\right| = \bar{o}\big(u^{-1/2}\big), \quad (27)$$

$t(u) = (t - mu)/(Du^{1/2})$, $Q_1(t) = \frac{1}{6}\chi_{(3,0)}(t^2 - 1) - \eta$, and $\chi_{(3,0)}$, η are certain constants found both in terms of ladder variables (see Malinovskii (1994), p. 165) and of Spitzer's sums (see Malinovskii (1994), p. 166). It was demonstrated that the approximation (27) coincides up to negligible terms with the Asmussen's in the "classical" case when the later is valid.

The author's conjecture is that the Edgeworth-like correction of (12) similar to (27) could be obtained by the price of a certain development of the technique introduced in Malinovskii (1994) and in Malinovskii (1997).

5.3 Large deviations

Both of the approximations (9) and (27), though proved to be uniform in $t \geqslant 0$, are designed for $t = t_u$ such that $(t_u - mu)/(Du^{1/2})$ remains bounded, as u grows. It means that (9) and (27) are expected to work satisfactorily only in $u^{1/2}$-neighborhoods of the line

$$N(t, u) = \{t > 0, u > 0 : t = mu\},$$

as $u \to \infty$.

The nature of these difficulties is well known. Indeed, the ruin after $t_u \gg mu$ and the ruin before $0 < t_u \ll mu$, as $u \to \infty$, constitute the so-called "large deviations" in the ruin problem. These events deserve a separate analysis known to depend crucially on a number of specific technical assumptions on the model.

The "large deviations" in the ruin problem were considered in a number of papers. E.g., Martin-Löf (1986) have derived upper bounds for $\psi_c(t, u)$ which could be applied for t and u positive and outside the $u^{1/2}$-neighborhood of $N(t, u)$. This result was obtained in the framework of the compound Poisson risk model and for the claim size distribution with exponential tails. Constructing an approximation for $\psi_c(t, u)$, Höglund (1990) took into account the large deviations. His basic assumption was that the moment generating function of the random vector (Y_1, T_1) is finite for some argument within the first quadrant of $\mathbf{R} \times \mathbf{R}$. The asymptotic expressions and the upper bounds on the probabilities of "large deviations" for heavy tailed interclaim random variables were obtained in Malinovskii (1994).

The author's conjecture is that the "large deviations" results similar to those obtained by Höglund (1990) and by Malinovskii (1996) in the framework of constant premium intensities, could be extended to the framework of variable ones.

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