$$+\frac{1}{4!\cdot(2z)^4}\left[n^8-21\,n^6+\frac{987}{8}n^4-\frac{3229}{16}n^2+\frac{11\,025}{256}\right]+\cdots\Bigg\}.$$

Proof. See Whittaker and Watson (1963), Section 17.7, item (vi), or Watson (1945), Section 7.23. □

For $a \geqslant 0$, $t > 0$ and $k = 1, 2, \ldots$, set

$$T_k(t,a)=\int_t^\infty \theta^{-(2k+1)/2}\exp\{-a\theta\}\,\mathrm{d}\theta.$$

Remark 6.1. One easily has

$$0\leqslant T_k(t,a)\begin{cases}\leqslant t^{-k-1/2}\displaystyle\int_t^\infty \mathrm{e}^{-a\theta}\mathrm{d}\theta=\frac{\mathrm{e}^{-at}}{a}t^{-k-1/2}, & a>0,\\ =\displaystyle\int_t^\infty \theta^{-k-1/2}\mathrm{d}\theta=\frac{2}{2k-1}t^{-k+1/2}, & a=0.\end{cases}$$

The following results elaborate the analysis of $T_k(t,a)$.

Lemma 6.4. *For $a \geqslant 0$, the recursion*

$$T_k(t,a)=(2k-1)^{-1}\left[2\sqrt{2\pi}\,t^{-k+1/2}\varphi_{\{0,1\}}(\sqrt{2at})-2aT_{k-1}(t,a)\right],\quad k=2,3,\ldots,$$

where $T_1(t,a) = 2t^{-1/2}\sqrt{2\pi}\varphi_{\{0,1\}}(\sqrt{2at}) - 2\sqrt{2\pi}a^{1/2}\{1 - \Phi_{\{0,1\}}(\sqrt{2at})\}$, holds true.

Proof. For $m_k(x) = \int_x^\infty z^{-k}\varphi_{\{0,1\}}(z)\mathrm{d}z$, the recursion

$$m_k(x)=\left(x^{-k+1}\varphi_{\{0,1\}}(x)-m_{k-2}(x)\right)/(k-1),\quad k=2,3,\ldots,$$

is easy to verify. Since $T_k(t,a) = 2^{k+1}\sqrt{\pi}a^{k-1/2}m_{2k}(\sqrt{2at})$, the proof is completed by direct algebra. □

The following corollary of Lemma 6.4 yields for $T_k(t,a)$ a closed-form expression instead of recursion.

Corollary 6.1. *For $a \geqslant 0$ and $k = 1, 2, \ldots$,*

$$T_k(t,a)=2\frac{\sqrt{2\pi}\varphi_{\{0,1\}}(\sqrt{2at})}{(2k-1)!!}\times\left\{\mathcal{P}_k(2at)+(-1)^k(\sqrt{2at})^{2k-1}M(\sqrt{2at})\right\}t^{-k+1/2},\tag{39}$$

where $M(x)$ is the Mill's ratio, $\mathcal{P}_1(x) = 1$ and $\mathcal{P}_k(x) = (2k-3)!! - x\,\mathcal{P}_{k-1}(x)$, $k = 2, 3, \ldots$, or in a closed form (set $(-1)!! = 1$)

$$\mathcal{P}_k(x)=(-1)^{k-1}x^{k-1}\sum_{m=0}^{k-1}(-1)^m(2m-1)!!x^{-m}.$$

In particular,

$$\mathcal{P}_1(x)=1,\qquad \mathcal{P}_2(x)=1-x,\qquad \mathcal{P}_3(x)=3!!-x+x^2,$$
$$\mathcal{P}_4(x)=5!!-3!!\,x+x^2-x^3,$$
$$\mathcal{P}_5(x)=7!!-5!!\,x+3!!\,x^2-x^3+x^4.$$

The following corollary of Lemma 6.4 yields expansions for $T_k(t,a)$ with $a > 0$ and is based on the application of Lemma 6.1 to Eq. (39).

Corollary 6.2. *For $a > 0$, $k = 1, 2, \ldots$ and for arbitrary integer $n > k$, one has*

$$T_k(t,a)=2\frac{\sqrt{2\pi}\varphi_{\{0,1\}}(\sqrt{2at})}{(2k-1)!!}(2a)^{k-1/2}\times\left\{\sum_{m=k}^{n-1}(-1)^{k+m}(2m-1)!!(2at)^{-m-1/2}+R_n(2at)\right\},$$

where $|R_n(2at)| < (2n-1)!!(2at)^{-n-1/2}$.

Lemma 6.5. *For u, b positive and $k = 0, 1, 2, \ldots$, one has*

$$S_k(u,b)=\mathrm{e}^{-u}\sum_{n\geqslant 0}\frac{u^n}{n!}b^{n+1}(n+1)^{k+1}=\exp\{-u(1-b)\}u^{-1}\Pi_{k+2}(bu),$$

where $\Pi_{k+2}(bu) = -\mathrm{i}^{k+2}\frac{\mathrm{d}^{k+2}}{\mathrm{d}t^{k+2}}\exp\{bu(\mathrm{e}^{\mathrm{i}t}-1)\}|_{t=0}$ is the $(k+2)$nd power moment of a Poisson random variable with parameter bu. One has $\Pi_{k+2}(bu) = bu\sum_{j=0}^{k+1}\binom{k+1}{j}\Pi_j(bu) = \sum_{j=1}^{k+2}S(k+2,j)(bu)^j$, where $S(m,n)$ are Stirling numbers of the second kind. In particular,

$$S_0(u,b)=b\exp\{-u(1-b)\}(1+bu),$$
$$S_1(u,b)=b\exp\{-u(1-b)\}(1+3\,bu+(bu)^2),$$
$$S_2(u,b)=b\exp\{-u(1-b)\}(1+7\,bu+6(bu)^2+(bu)^3),$$
$$S_3(u,b)=b\exp\{-u(1-b)\}(1+15\,bu+25(bu)^2+10(bu)^3+(bu)^4),$$
$$S_4(u,b)=b\exp\{-u(1-b)\}(1+31\,bu+90(bu)^2+65(bu)^3+15(bu)^4+(bu)^5).$$

Proof. The proof is straightforward. □

Acknowledgement

The author is grateful to the anonymous referee for many constructive remarks and suggestions which led to substantial improvement in the presentation of this paper.

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