Bearing further in mind that $\Gamma(k+\frac{1}{2}) = \sqrt{2\pi}(k+\frac{1}{2})^k \,\mathrm{e}^{-k-1/2}(1+\bar{o}(1)), k! = \sqrt{2\pi k}\, k^k \mathrm{e}^{-k} \times (1+\bar{o}(1))$, so that $\Gamma(k+\frac{1}{2})/k! = k^{-1/2}(1+\bar{o}(1))$, and that $\zeta(k+\frac{1}{2}) \to 1$, as $k \to \infty$, we easily obtain the relation (6.2) for $l = 1$. It is worth noting that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(k+\frac{1}{2}) = \sqrt{\pi}\frac{1}{2}\dots(k-\frac{3}{2})(k-\frac{1}{2}),\ k = 1, 2, \dots$.

To obtain (6.2) for $l = 3$, note that termwise differentiation is allowed (see Whittaker and Watson (1963), Chap. II, Section 2.61) and obtain for sufficiently small a

$$\sum_{n=1}^{\infty} \sqrt{n}\,\mathrm{e}^{-an} = \frac{\sqrt{\pi}}{2a^{3/2}} - \sqrt{\frac{1}{\pi}} \sum_{k=1}^{\infty} \frac{\Gamma(k+\frac{1}{2})\zeta(k+\frac{1}{2})}{(2\pi)^k (k-1)!} \omega_k a^{k-1}.$$

Further termwise differentiation of (6.3) completes the proof.

Lemma 6.8. *The inequality* $(a+b)^p \leqslant 2^{p-1}(a^p+b^p)$ *holds true for* $a, b > 0$ *and* $p \geqslant 1$.

Proof of Lemma 6.8. Put $f(x) = (a+x)^p - 2^{p-1}(a^p+x^p)$. Since $f'(x) = p(a+x)^{p-1} - 2^{p-1}px^{p-1}$ and $p \geqslant 1$, $f'(a) = 0$, $f'(x) > 0$, as $x < a$, and $f'(x) < 0$, as $x > a$. Therefore $f(b) \leqslant \max_x f(x) = f(a) = 0$ and the proof is complete.

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