MANAGING SOLVENCY — A RISK THEORY INSIGHT

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AGENDA

1. **Introduction: a deficiency of traditional Risk Theory**
2. **Managing solvency: simulation analysis of insurance risk process, scenario-based DFA, directives**
3. **Modelling of multiperiodic controlled insurance process**
4. **Synthesis of adaptive control rules in generic models**
5. **Performance of adaptive control strategies**

1. Introduction: a deficiency of traditional Risk Theory

References

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[2] Borch, K. (1967) The theory of risk, *Journal of the Royal Statist. Soc., Ser. B*, vol. 29, no. 3, 432–452; Discussion, *ibid.*, 452–467.
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• Quotation from **H. Bohman'** (p. 2) farewell interview as retiring Chief Editor of the Scandinavian Actuarial Journal:

"I was for a long time deeply involved in this theory, working on the probability of ruin, but I am hesitant over it now . . . From a practical point of view, the theory of collective risk, as initiated by Filip Lundberg, has missed the point, *because the underlying model is unrealistic, too simplified*. For one thing, a stationary business should give stationary reserves, as predicted by the control theory."

• Quotation from **H. Cramér**:

"In view of certain misconceptions that have appeared it is, however, necessary to point out that *Lundberg repeatedly emphasizes the practical importance of some arrangement which automatically prevents the risk reserve from growing unduly*. This point is, in fact, extensively discussed in the papers of 1909, 1919 and 1926 − 28. One possible arrangement proposed to this end is to work with a security factor $\tau = \tau(x)$ which is a decreasing function of the risk reserve $R(t) = x$. Another possibility is to dispose, at predetermined epochs, of part of the risk reserve for bonus distribution. By either method, the growth of the risk reserve may be efficiently controlled. What Lundberg does in this connection is really to work with a rather refined case of what has much later come to be known as a random walk with two barriers.

From certain quarters, the Lundberg's theory has been declared to be unrealistic because, it is asserted, no limit is imposed on the growth of the risk reserve. In view of what has been said above, it would seem that these critics have not read the author they are criticizing. For a non-Scandinavian author there is, of course, the excuse that most of Lundberg's works are written in Swedish."

- Quotation from **C. Philipson** (p. 68):

"From the development of the classical form [of the risk theory. — V.M.] two lines of development have branched out, one refers to the *generalization of the fundamental assumptions*... The other refers to the *extensions of the decision theory*... These lines of development are, however, all based on the fundamental conception of the collective risk theory, which was created by Filip Lundberg..."

- Quotation from **K. Borch** (p. 451):

"We have now reached the point where the actuarial theory of risk again joins the mainstream of theoretical statistics and applied mathematics. Our general formulation of the actuary's problem leads directly to the general theory of *optimal control processes* or *adaptive control processes*...

The theory of control processes seems to be "tailor–made" for the problems which actuaries have struggled to formulate for more than a century."

• Quotation from **C.D. Daykin, T. Pentikäinen, M. Pesonen** (Chapter 1, Section 5.5, p. 154):

"It is worth mentioning that *the classical analytical methods and simulation should not be regarded as being in competition*. A general rule is that an *analytical technique should always be used wherever it is tractable*. On the other hand, the temptation should be resisted to manipulate the premises of the model in order to make the analytical calculations possible, if that can only be done at the cost of the applicability of the model to real–world conditions. If that is done, as is often the case in theoretically–orientated risk theory, a warning of the restricted applicability — or non–applicability — should be clearly given. The wide realm of application of simulation methods begins at the frontier where other methods become intractable."

2. Managing solvency: simulation analysis of insurance risk process, scenario-based DFA, directives

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3. Modelling of multiperiodic controlled insurance process

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Multiperiodic controlled insurance process

General multiperiodic insurance process with annual accounting, built-in information system and annual control interventions

$$\underbrace{\begin{pmatrix}\mathfrak{w}_0\\ \mathfrak{s}_0\end{pmatrix} \xrightarrow{\gamma_0} \mathfrak{u}_0 \xrightarrow{\pi_1} \begin{pmatrix}\mathfrak{w}_1\\ \mathfrak{s}_1\end{pmatrix}}_{\text{1-st year}} \cdots \xrightarrow{\pi_{k-1}} \begin{pmatrix}\mathfrak{w}_{k-1}\\ \mathfrak{s}_{k-1}\end{pmatrix} \underbrace{\xrightarrow{\gamma_{k-1}} \mathfrak{u}_{k-1} \xrightarrow{\pi_k} \begin{pmatrix}\mathfrak{w}_k\\ \mathfrak{s}_k\end{pmatrix}}_{k\text{-th year}} \cdots .$$

Singleperiodic (annual generic) risk models

- *Diffusion*
- *Poisson–Exponential (classical)*

Scenarios of Nature (Completely Known Nature Scenario, Stable Nature Scenario, Fluctuating Nature Scenario, etc.)

Singleperiodic (annual generic) Poisson–Exponential risk model

Poisson–Exponential (or classical) risk model: the risk reserve at time t is

$$R_s(u,c,\tau) = u + c(1+\tau)s - V_s, \quad V_s = \sum_{i=1}^{N(s)} Y_i, \quad 0 \leqslant s \leqslant t,$$

where u is the initial risk reserve, c is the risk premium rate, τ is the adaptive premium loading, t is the year duration, $\{T_i\}_{i\geqslant 1}$ and $\{Y_i\}_{i\geqslant 1}$ are i.i.d. and mutually independent, where T_i are the interclaim times and Y_i are the amounts of claims, exponentially distributed with parameters $\lambda > 0$ and $\mu > 0$, respectively, $N(t)$ is the largest n for which $\sum_{i=1}^{n} T_i \leqslant t$ (we put $N(t) = 0$ if $T_1 > t$). Put $M_t(u,c,\tau) = \inf_{0<s\leqslant t} R_s(u,c,\tau)$.

Note that

$$\mathsf{E}V_s = \frac{\lambda}{\mu}\, s, \quad s \geqslant 0,$$

so the premium rate $c = \dfrac{\lambda}{\mu}$ (when λ and μ are known) is calculated according to the Equity (Expected value) Principle.

Singleperiodic (annual generic) diffusion risk model

Diffusion risk model:

$$R_s(u,c,\tau) = u + c(1+\tau)s - V_s, \quad V_s = \mu s + \sigma \mathsf{W}_s, \quad 0 \leqslant s \leqslant t,$$

where u is the initial risk reserve, c is the risk premium rate, τ is the adaptive premium loading, t is the year duration, V_s, $0 \leqslant s \leqslant t$, is the claims out-pay process with claims out-pay rate μ, diffusion coefficient $\sigma > 0$ and standard Brownian motion W_s, $0 \leqslant s \leqslant t$. Put

$$M_t(u,c,\tau) = \inf_{0<s\leqslant t} R_s(u,c,\tau).$$

The couple (R_t, M_t) is taken generic for the *state variable* which describes insurer's annual financial experience.

The triplet (u,c,τ) generates three-dimensional *control variable*.

Rigorous definition of a model and synthesis of the adaptive control rules satisfying desirable performance criteria is the central problem. It is tightly connected with the Scenarios of Nature.

Scenarios of Nature (for diffusion generic risk models)

Completely Known Nature Scenario

For $k = 1, 2, \ldots$, all claims out-pay rate parameters $\mu_k > 0$ and diffusion parameters $\sigma_k > 0$ of the annual claims out-pay processes $V_s^{[k]}$, $0 \leqslant s \leqslant t$, are known in advance.

Stable Nature Scenario

For $k = 1, 2, \ldots$, claims out-pay rate parameters $\mu_k > 0$ and diffusion parameters $\sigma_k > 0$ remain unchanged for years, $\mu_1 = \mu_2 = \cdots = \mu$ and $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma^2$ (to be specific, assume that $\mu \in \mathrm{M} = [a, b] \subset \mathsf{R}^+$), σ^2 is known but μ is unknown.

Fluctuating Nature Scenario

The claims out-pay rate parameters μ_k, $k = 1, 2, \ldots$, are i.i.d. Normal random variables with known mean μ and variance $s_t > 0$, and independent on the insurance process. The diffusion parameters σ_k, $k = 1, 2, \ldots$, are known positive numbers.

Transition functions π_k in multiperiodic controlled risk model (for diffusion generic risk models and Stable Nature Scenario)

The background data is a series ($k = 1, 2, \dots$) of mutually independent standard Brownian motions $\mathsf{W}_s^{[k]}$, $0 \leqslant s \leqslant t$. Put

$$\mathrm{W} = \mathsf{R} \times \{0, 1\}, \quad \mathrm{M} = [a, b], \quad \mathrm{U} = \mathsf{R}^+ \times \mathsf{R}^+ \times \mathsf{R}$$

and specify t.f.p.m. π_k, $k = 1, 2, \dots$. For $k = 1, 2, \dots$ introduce $V_s^{[k]} = \mu s + \sigma \mathsf{W}_s^{[k]}$, $0 \leqslant s \leqslant t$, set $\mathfrak{u}_{k-1} = (\mathfrak{u}_{k-1}^{\langle 1 \rangle}, \mathfrak{u}_{k-1}^{\langle 2 \rangle}, \mathfrak{u}_{k-1}^{\langle 3 \rangle}) \in \mathrm{U}$ and consider

$$\begin{aligned} R_s^{[k]}(\mathfrak{u}_{k-1}) &= \mathfrak{u}_{k-1}^{\langle 1 \rangle} + \mathfrak{u}_{k-1}^{\langle 2 \rangle}(1 + \mathfrak{u}_{k-1}^{\langle 3 \rangle}) s - V_s^{[k]}, \\ M_t^{[k]}(\mathfrak{u}_{k-1}) &= \inf_{0 < s \leqslant t} R_s^{[k]}(\mathfrak{u}_{k-1}), \\ \hat{\mu}_t^{[k]} &= t^{-1} V_t^{[k]} = \mu + \sigma t^{-1} \mathsf{W}_t^{[k]} \qquad \text{(Maximum likelihood estimate, MLE).} \end{aligned}$$

For $k = 1, 2, \dots$ set $\mathfrak{w}_k = (\mathfrak{w}_k^{\langle 1 \rangle}, \mathfrak{w}_k^{\langle 2 \rangle}) \in \mathrm{W}$, $\mathfrak{s}_k \in \mathrm{M}$ and introduce t.f.p.m.

$$\begin{aligned} &\pi_k(\mathfrak{w}_{k-1}, \mathfrak{s}_{k-1}, \mathfrak{u}_{k-1}; d\mathfrak{w}_k \times d\mathfrak{s}_k) \\ &\qquad = \mathsf{P}_\mu\{R_t^{[k]}(\mathfrak{u}_{k-1}) \in d\mathfrak{w}_k^{\langle 1 \rangle}, \mathbf{1}_{\{M_t^{[k]}(\mathfrak{u}_{k-1}) < 0\}} \in d\mathfrak{w}_k^{\langle 2 \rangle}, \hat{\mu}_t^{[k]} \in d\mathfrak{s}_k\} \quad \text{for } \mu \in \mathrm{M}. \end{aligned}$$

Control strategies in multiperiodic controlled risk model

Set $d_\varepsilon = \Phi^{-1}_{\{0,1\}}(1-\varepsilon) \geqslant 0$ for $0 < \varepsilon \leqslant 1/2$ ($d_{1/2} = 0$) and note that for

$$\hat{\upsilon}^{[k]}_{t,\varepsilon} = \hat{\mu}^{[k]}_t + \sigma t^{-1/2} d_\varepsilon,$$

$$\mathsf{P}_\mu\{\mu < \hat{\upsilon}^{[k]}_{t,\varepsilon}\} = 1-\varepsilon \quad \text{for all } \mu \in [a,b].$$

Say for $0 < \varepsilon < 1/2$ that μ is overestimated by $\hat{\upsilon}^{[k]}_{t,\varepsilon}$, or ε-overestimated.

- *A Markov strategy:* $\gamma_{k-1}(\cdot) : \mathrm{W} \times \mathrm{M} \to \mathrm{U}$, $k = 1, 2, \ldots$

For $0 < \alpha < 1$, $0 < \varepsilon \leqslant 1/2$, the initial control $(\mathfrak{u}^{\langle 1\rangle}_0, \mathfrak{u}^{\langle 2\rangle}_0, \mathfrak{u}^{\langle 3\rangle}_0)$ and the sequence $(\mathfrak{u}^{\langle 1\rangle}_k, \mathfrak{u}^{\langle 2\rangle}_k, \mathfrak{u}^{\langle 3\rangle}_k)$, $k = 1, 2, \ldots$, where

$$\mathfrak{u}^{\langle 1\rangle}_k = \mathfrak{w}^{\langle 1\rangle}_k,\ \mathfrak{u}^{\langle 2\rangle}_k = \mathfrak{s}_k + \sigma t^{-1/2} d_\varepsilon,\ \mathfrak{u}^{\langle 3\rangle}_k = -\frac{z}{t(\mathfrak{s}_k + \sigma t^{-1/2} d_\varepsilon)}$$

with $z = \mathfrak{w}^{\langle 1\rangle}_k - u_{d_\varepsilon}(\alpha, t, 1)$, *deviation of the past-year-end risk reserve* (i.e., a financial result of the previous year) *from* $u_0(\alpha, t, 1)$, is called Markov basic adaptive control strategy with ε-<u>overestimated</u> μ. When $\varepsilon = 1/2$ (which implies $d_\varepsilon = 0$), it is called Markov basic adaptive control strategy with <u>estimated</u> μ.

- *A zone-adaptive non-Markov strategy:* $\gamma_{k-1}(\cdot) : (\mathrm{W} \times \mathrm{M})^k \to \mathrm{U}$, $k = 1, 2, \ldots$

For $0 < \alpha \leqslant \beta < 1$, $0 < \varepsilon \leqslant 1/2$, the initial control $(\mathfrak{u}_0^{\langle 1\rangle}, \mathfrak{u}_0^{\langle 2\rangle}, \mathfrak{u}_0^{\langle 3\rangle})$ and the sequence $(\mathfrak{u}_k^{\langle 1\rangle}, \mathfrak{u}_k^{\langle 2\rangle}, \mathfrak{u}_k^{\langle 3\rangle})$, $k = 1, 2, \ldots$, where

$$\mathfrak{u}_k^{\langle 1\rangle} = \begin{cases} u_{t,k}^{\mathrm{low}}(\alpha, \beta, \varepsilon), & z < z_{t,k}^{\mathrm{low}}(\alpha, \beta, \varepsilon), \\ \mathfrak{w}_k^{\langle 1\rangle}, & z_{t,k}^{\mathrm{low}}(\alpha, \beta, \varepsilon) \leqslant z \leqslant 0, \\ u_{d_\varepsilon}(\alpha, t, k), & z > 0, \end{cases}$$

$$\mathfrak{u}_k^{\langle 2\rangle} = \mathfrak{S}_k,$$

$$\mathfrak{u}_k^{\langle 3\rangle} = \begin{cases} \tau_{t,k}^{\mathrm{low}}(\alpha, \beta, \varepsilon), & z < z_{t,k}^{\mathrm{low}}(\alpha, \beta, \varepsilon), \\ -\dfrac{z}{t\mathfrak{S}_k}, & z_t^{\mathrm{low}}(\alpha, \beta, \varepsilon) \leqslant z \leqslant 0, \\ 0, & z > 0 \end{cases}$$

with $z = \mathfrak{w}_k^{\langle 1\rangle} - u_{d_\varepsilon}(\alpha, t, k)$, $\mathfrak{S}_k = k^{-1} \sum_{i=1}^{k} \mathfrak{s}_i + \sigma(tk)^{-1/2} d_\varepsilon$, is called non-Markov zone-adaptive control strategy with ε-overestimated μ. In the particular case $\varepsilon = 1/2$ (which implies $d_\varepsilon = 0$) it is called non-Markov zone-adaptive control strategy with estimated μ.

Rigorous mathematical model

By multiperiodic controlled insurance process with built-in information system we mean the controlled random sequence

$$(W_k, S_k, U_k), \quad k = 0, 1, \dots,$$

defined over the probability space $(\Omega, \mathcal{F}, \mathbf{P}_\mu^{\pi,\gamma})$ with $\mu \in \mathrm{M}$. The random variables (W_k, S_k), $k = 0, 1, \dots,$ with realizations $(\mathfrak{w}_k, \mathfrak{s}_k)$ assume values in the output space $(\mathrm{W} \times \mathrm{M}, \mathcal{W} \otimes \mathcal{M})$ which is a product of the financial experience state space and statistical information state space, the random variables U_k, $k = 0, 1, \dots,$ with realizations $\mathfrak{u}_k$ assume values in the input, or control state space $(\mathrm{U}, \mathcal{U})$.

For each $\mu \in \mathrm{M}$, the measure $\mathbf{P}_\mu^{\pi,\gamma}$ is defined over the elementary state space $(\Omega, \mathcal{F})$ as a unique probability measure such that for every rectangle $A_0 \times \cdots \times A_n \times B_0 \times \cdots \times B_{n-1}$

$$\begin{aligned} &\mathbf{P}_\mu^{\pi,\gamma}\{(W_0, S_0) \in A_0, U_0 \in B_0, \dots, U_{n-1} \in B_{n-1}, (W_n, S_n) \in A_n\} \\ &\qquad = \int_{A_0} \pi_0(d\mathfrak{w}_0 \times d\mathfrak{s}_0) \int_{B_0} \gamma_0((\mathfrak{w}_0, \mathfrak{s}_0); d\mathfrak{u}_0) \int_{A_1} \pi_1(\mathcal{Y}_0, \mathfrak{u}_0; d\mathfrak{w}_1 \times d\mathfrak{s}_1) \dots \\ &\qquad\qquad \dots \int_{B_{n-1}} \gamma_{n-1}(\mathcal{Y}_{n-1}; d\mathfrak{u}_{n-1}) \int_{A_n} \pi_n(\mathcal{Y}_{n-1}, \mathfrak{u}_{n-1}; d\mathfrak{w}_n \times d\mathfrak{s}_n), \end{aligned}$$

where $n \in \mathsf{N}$, $A_k \in \mathcal{W} \otimes \mathcal{M}$, $B_k \in \mathcal{U}$, $k = 0, 1, \dots,$ $\mathcal{Y}_0 = \{\mathfrak{w}_0, \mathfrak{s}_0\}$ and

$$\mathcal{Y}_{k-1} = \{(\mathfrak{w}_0, \mathfrak{s}_0), \dots, (\mathfrak{w}_{k-1}, \mathfrak{s}_{k-1}), \mathfrak{u}_0, \dots, \mathfrak{u}_{k-2}\}.$$

4. Synthesis of adaptive control rules in generic models

Diffusion risk model:

$$R_s(u,c,\tau) = u + c(1+\tau)s - V_s, \quad V_s = \mu s + \sigma \mathsf{W}_s, \quad 0 \leqslant s \leqslant t,$$

where u is the initial risk reserve, c is the risk premium rate, τ is the adaptive premium loading, t is the year duration, V_s, $0 \leqslant s \leqslant t$, is the claims out-pay process with claims out-pay rate μ, diffusion coefficient $\sigma > 0$ and standard Brownian motion W_s, $0 \leqslant s \leqslant t$. Put

$$M_t(u,c,\tau) = \inf_{0<s\leqslant t} R_s(u,c,\tau).$$

ASSUMPTION 1. For every integer k, there exists a Normal random variable $\hat{\mu}_{tk}$ independent on Brownian motion W_s, $0 \leqslant s \leqslant t$, such that $\mathsf{E}_\mu \hat{\mu}_{tk} = \mu$ and $\mathsf{D}_\mu \hat{\mu}_{tk} = \sigma^2 (tk)^{-1}$ for all $\mu \in \mathrm{M} = [a,b]$.

DEFINITION 1. For k integer, $0 < \alpha < 1$, the *target capital value* $u_{d_\varepsilon}(\alpha,t,k)$ corresponding to the random risk premium rate $\hat{\upsilon}_{tk,\varepsilon} = \hat{\mu}_{tk} + \sigma(tk)^{-1/2} d_\varepsilon$ is a positive solution of the equation

$$\mathsf{P}_\mu\{M_t(u, \hat{\upsilon}_{tk,\varepsilon}, 0) < 0\} = \alpha.$$

THEOREM 1. For k integer, $0 < \alpha < 1$, $0 < \varepsilon \leqslant 1/2$ and $d_\varepsilon = \Phi^{-1}_{\{0,1\}}(1-\varepsilon)$, the solution of equation

$$\mathsf{P}_\mu\{M_t(u, \hat{\upsilon}_{tk,\varepsilon}, 0) < 0\} = \alpha,$$

where $\hat{\upsilon}_{tk,\varepsilon} = \hat{\mu}_{tk} + \sigma(tk)^{-1/2} d_\varepsilon$, is

$$u_{d_\varepsilon}(\alpha, t, k) = \sigma\sqrt{t}\, y_k,$$

and y_k is a positive solution of the equation

$$\Phi_{\{0,1\}}\Big(-\frac{y\sqrt{k}+d_\varepsilon}{\sqrt{k+1}}\Big) + \exp\Big\{-2y\big(\frac{d_\varepsilon}{\sqrt{k}} - \frac{y}{k}\big)\Big\}\Phi_{\{0,1\}}\Big(-\frac{y(k+2)-d_\varepsilon\sqrt{k}}{\sqrt{k(k+1)}}\Big) = \alpha.$$

TABLE 1. Values of $u_{d_\varepsilon}(\alpha, t, 1)$ for $t = 100$, $\sigma = 1$ calculated numerically.

ε	$d_\varepsilon = \Phi^{-1}_{\{0,1\}}(1-\varepsilon)$	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.15$	$\alpha = 0.2$
0.5	0	25.4677	20.7965	17.7532	15.4216
0.3	0.5244	20.7912	16.3664	13.5647	11.4824
0.1	1.2816	14.6251	10.8657	8.66394	7.13512
0.05	1.6449	12.0769	8.78949	6.94727	5.70197

For k integer, $0 < \varepsilon \leqslant 1/2$, $0 < \alpha < 1$ and $z \in \mathsf{R}$, for triplet

$$u_{z,t,k} = u_{d_\varepsilon}(\alpha, t, k) + z, \quad c_{t,k} = \hat{\upsilon}_{tk,\varepsilon}, \quad \tau_{z,t,k} = -\frac{z}{t\hat{\upsilon}_{tk,\varepsilon}}$$

set

$$\psi_{t,k,\alpha,\varepsilon}(z) = \mathsf{P}_\mu\{M_t(u_{z,t,k}, c_{t,k}, \tau_{z,t,k}) < 0\}.$$

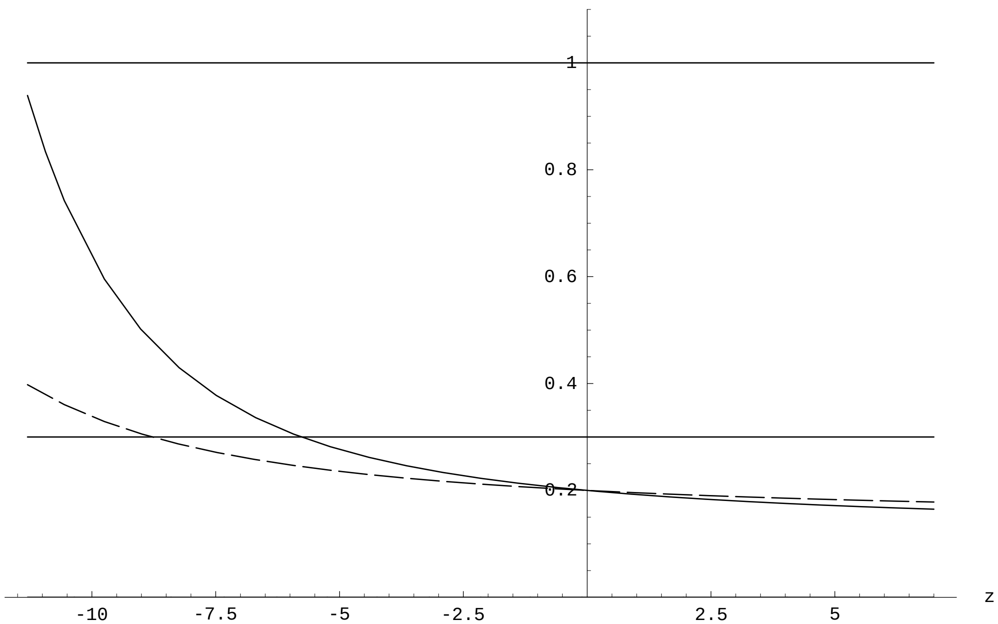

Figure 1: $\psi_{t,1,\alpha,\varepsilon}(z)$ for $\varepsilon = 0.3$, $d_\varepsilon = 0.5244$, $\alpha = 0.2$, $t = 100$, $\sigma = 1$, $u_{d_\varepsilon}(\alpha, t, 1) = 11.4824$ (solid line), for $\varepsilon = 0.5$, $d_\varepsilon = 0$, $\alpha = 0.2$, $t = 100$, $\sigma = 1$, $u_{d_\varepsilon}(\alpha, t, 1) = 15.4216$ (dashed line), and the level $\beta = 0.3$.

THEOREM 2. For k integer, $0 < \varepsilon \leqslant 1/2$, $0 < \alpha \leqslant \beta < 1$, $z \in (-u_{d_\varepsilon}(\alpha, t, k), \infty)$ and for triplet

$$u_{z,t,k} = u_{d_\varepsilon}(\alpha, t, k) + z, \quad c_{t,k} = \hat{\upsilon}_{tk,\varepsilon}, \quad \tau_{z,t,k} = -\frac{z}{t\hat{\upsilon}_{tk,\varepsilon}}$$

the equation

$$\mathsf{P}_\mu\{M_t(u_{z,t,k}, c_{t,k}, \tau_{z,t,k}) < 0\} = \beta \qquad (\text{i.e.} \quad \psi_{t,k,\alpha,\varepsilon}(z) = \beta)$$

has exactly one non-positive root

$$z_{t,k}^{\text{low}}(\alpha, \beta, \varepsilon) \leqslant 0$$

with the equality sign if and only if $\alpha = \beta$.

DEFINITION 2. The value

$$u_{t,k}^{\text{low}}(\alpha, \beta, \varepsilon) = u_{d_\varepsilon}(\alpha, t, k) + z_{t,k}^{\text{low}}(\alpha, \beta, \varepsilon)$$

is called *lower bound of a strip zone* corresponding to levels α and β.

5. Performance of adaptive control strategies

Targeting (corresponds to the Equity Principle)

Targeting means ability to keep year-by-year the risk reserve process inside a strip zone centered around a target value.

Solvency (corresponds to the Solvency Principle)

Solvency means existence of satisfactory upper bounds on the probability of ruin within each insurance year and within a sequence of successive insurance years.

Dynamic solvency provisions

Dynamic solvency provisions are those funds which are accumulated in the profitable years, i.e. when the year-end risk reserve exceeds the upper bound of the strip zone, and consumed in the deficiency years, i.e. when the year-end risk reserve felt below the lower bound of the strip zone.

Targeting

THEOREM 3. For $0 < \alpha < 1$, $0 < \varepsilon \leqslant 1/2$, for multiperiodic controlled insurance process agreed with Stable Nature Scenario and Markov strategy, all mean year-end capitals equal to $u_{d_\varepsilon}(\alpha, t, 1) + \sigma t^{1/2} d_\varepsilon$, i.e.

$$\mathbf{E}_\mu^{\pi;\gamma} W_k^{\langle 1\rangle} = u_{d_\varepsilon}(\alpha, t, 1) + \sigma t^{1/2} d_\varepsilon$$

for all $\mu \in \mathrm{M} = [a, b]$ and $k = 1, 2, \ldots$.

In particular, for $\varepsilon = 1/2$,

$$\mathbf{E}_\mu^{\pi;\gamma} W_k^{\langle 1\rangle} = u_0(\alpha, t, 1)$$

for all $\mu \in \mathrm{M} = [a, b]$ and $k = 1, 2, \ldots$.

Similar result holds true for zone-adaptive non-Markov strategy.

Solvency

THEOREM 4. For $0 < \alpha \leqslant \beta < 1$, $0 < \varepsilon \leqslant 1/2$, for multiperiodic controlled insurance process agreed with Stable Nature Scenario and zone-adaptive non-Markov strategy

$$\mathbf{P}_{\mu}^{\pi,\gamma}\{\text{first ruin in year } k\} \leqslant \beta$$

for all $\mu \in \mathrm{M} = [a, b]$ and $k = 1, 2, \ldots$.

THEOREM 5. In the assumptions of Theorem 4, for each integer n

$$\mathbf{P}_{\mu}^{\pi,\gamma}\{\text{ruin within } n \text{ years}\} = \sum_{k=1}^{n} \mathbf{P}_{\mu}^{\pi,\gamma}\{\text{first ruin in year } k\} \leqslant n\beta.$$

Dynamic solvency provisions

Definition 3. For $0 < \alpha \leqslant \beta < 1$, $0 < \varepsilon \leqslant 1/2$, for multiperiodic controlled insurance process agreed with Stable Nature Scenario and zone-adaptive non-Markov strategy, the variable

$$\Delta_t(\mathfrak{w}_k^{\langle 1\rangle}) = \begin{cases} 0, & u_{t,k}^{\mathrm{low}}(\alpha,\beta,\varepsilon) \leqslant \mathfrak{w}_k^{\langle 1\rangle} \leqslant u_{d_\varepsilon}(\alpha,t,k), \\ \mathfrak{w}_k^{\langle 1\rangle} - u_{d_\varepsilon}(\alpha,t,k), & \mathfrak{w}_k^{\langle 1\rangle} > u_{d_\varepsilon}(\alpha,t,k), \\ -(u_{t,k}^{\mathrm{low}}(\alpha,\beta,\varepsilon) - \mathfrak{w}_k^{\langle 1\rangle}), & \mathfrak{w}_k^{\langle 1\rangle} < u_{t,k}^{\mathrm{low}}(\alpha,\beta,\varepsilon) \end{cases}$$

is called k-th *year excess* (of either sign) *of capital*.

Theorem 6. For $0 < \alpha \leqslant \beta < 1$, $0 < \varepsilon \leqslant 1/2$, for multiperiodic controlled insurance process agreed with Stable Nature Scenario and zone-adaptive non-Markov strategy

$$\mathbf{E}_\mu^{\pi,\gamma} \Delta_t(W_k^{\langle 1\rangle}) > 0$$

for all $\mu \in \mathrm{M} = [a,b]$ and $k = 1, 2, \ldots$.