SURVIVE A DOWNSWING PHASE OF THE UNDERWRITING CYCLE

Vsevolod K. Malinovskii

Abstract. Profits in property and liability insurance tend to rise and fall in fairly regular patterns lasting between five and seven years from peak to peak; this phenomenon is termed the underwriting cycle. For a particular insurer, the cycles may be caused by difference between the price prevailing in the market and price of the risk in the insurer's portfolio. In this paper a multiperiod Lundberg-type control model of insurer's response to the cycle generated by the competition is developed.

1. Introduction

It is largely recognized (see e.g., [**25**]) that the long-term variations called "business cycles", are typically common for the most insurers and have several potential causes. There exists convincing evidence that the cycles are a fundamentally characteristic feature in most non-life business likely in all countries of competitive insurance market.

Understanding the driving forces of the underwriting cycles is a paramount theoretical problem, a key for understanding the nature of this phenomenon and a leverage for rational management. It attracts constant attention of many parties, including managers and experts in economical and actuarial studies.

There exist at least two major interpretations of the cyclic behavior in insurance. One ascribes the cycles to the fluctuations due to random surroundings, to volatile interest rates, or to random up- and down-swings of the risk exposure in the portfolio. Typically, such fluctuations can not be foreseen and their dynamics is known deficiently since its origin used to be exogenous with respect to the insurance industry. It causes inevitable errors in the rate making, and irregularly cyclic underwriting process ensues.

The other assigns the cycles to the strategies of aggressive insurers seeking for greater market shares, and by the consequent industry response. At the first stage, the response consists in concerted reduction of the rates, sometimes below the real costs of insurance. This makes some companies ruined, and agrees with the observation that insurance cycles are correlated with clustered insolvencies. For instance (see [**14**] with reference on Best's Insolvency Study [**4**]), US industry-wide combined ratios peaked at 109% in 1975 and 117% in 1984. The insurance failure rate, or the ratio of insolvencies to total companies, peaked at 1.0% in 1975 and 1.4% in 1985. Insolvencies appear a driving force behind the competition originated cycles since after elimination of the exceedingly aggressive and unwise agents, or just weaker carriers, the prices increase uniformly over the industry and the upswing phase of the cycle follows.

Key words and phrases. Underwriting cycles, Years of soft and hard market, Multiperiod insurance process, Solvency, Adaptive control strategies.

As underwriting cycles are due to competition, their downswing phases can have a particularly significant impact on the financial strength of each property and liability insurer who may be either ruined, having risk reserve exhausted because of too low price, or loose his business, having clients switched to those who have better prices. All that deserves thorough quantitative studies.

In the business and economical literature valuable contributions concerning insurance cycles abound. Among so many issues exploring the driving forces and the development of the cycles are [**28**], [**6**], [**11**], [**18**], [**10**], [**14**]. The latter publication is a convictive insight into the two different interpretations of the cycles in insurance; it provides as many as 109 references, yielding a comprehensive account of the modern state-of-art in the field.

There is however much less in the risk theory literature providing tools to quantify the additional risk associated with the underwriting cycles. Pentikäinen (see e.g., [**25**]) and then Daykin et al. [**9**] study relationship between the underwriting cycles and the ruin probabilities using mostly simulation techniques. Under the title of “dynamic financial analysis” that approach is further developed (see, e.g., [**8**], [**17**]). Providing conclusions drawn on the basis of empirical data, Rantala [**26**] discussed some potential background factors in cycle fluctuations and applied control–theoretical tools in the framework of autoregressive models. Seeking for a model of competition originated underwriting cycles, Feldblum [**14**] models the cycle by assuming that the risk loading and the claim rate follow a non-random trigonometric function. His surplus model allows the insurer to vary the price in response to the cycles, loosing or acquiring the market share. More complex theoretical insight into the general periodic behavior of the risk reserve process may be found in Asmussen and Rolski [**3**] who applied a Markovian model. From the premises of the individual approach, Subramanian [**29**] addressed solvency and market share balance while competition in the framework of a bonus–malus system.

The objective of the present paper is to contribute to quantitative analysis of the risk associated with the underwriting cycles. It develops the dynamic approach of [**21**]–[**24**]. While in [**21**] and [**24**] emphasis is made on harmonization of the equity and solvency requirements, i.e. on the correct rate making procedures, and further on deficiencies introduced by the exterior ambiguities limited by the so-called scenarios of nature, the present paper accentuates the cycles generated by competition.

As in [**21**] and [**24**], we do not intend to analyze in the present paper the real economical mechanism of the underwriting cycles. Rather, we focus on the question of how an insurance company can model and overcome the downswing phase of the cycle given that it occurs in consequence of a series of k underwriting years of severe price competition, called years of hard market, implemented in a series of descending market prices $P_1^M > \cdots > P_k^M > 0$, all below the average risk $\mathsf{E}Y$ of a particular insurer.

Since the period k of the underwriting cycle is known to be about six years, sensible is to analyze the cyclic evolution by means of a dynamic multiperiod model. Such a model, allowing for annual accountings and annual controls, was considered in [**21**]–[**24**]. It is described as

$$\underbrace{\mathfrak{w}_0 \xrightarrow{\gamma_0} \mathfrak{u}_0 \xrightarrow{\pi_1} \mathfrak{w}_1}_{\text{1st year, } P_1^M, \alpha_1} \;\cdots\; \xrightarrow{\pi_{k-1}} \mathfrak{w}_{k-1} \underbrace{\xrightarrow{\gamma_{k-1}} \mathfrak{u}_{k-1} \xrightarrow{\pi_k} \mathfrak{w}_k}_{k\text{th year, } P_k^M, \alpha_k} \;\cdots, \qquad (1)$$

where $\mathfrak{w}_{k-1}$ is the state variable observed at the end of $(k-1)$th year, γ_{k-1} is the control rule yielding the control variable $\mathfrak{u}_{k-1}$ at the beginning of kth year, π_k is the probability mechanism of insurance in kth year, α_k is the ruin level allowed in kth year. In [**21**]–[**24**] diverse π_k and γ_k were analyzed.

The paper is organized as follows. In Sections 2 and 3, we define the years of soft and hard market and discuss the dynamics of portfolio size. Annual risk reserve process are discussed in Section 4. Emphasized is underwriting, while investment, dividends and expenses components of the surplus process are ignored. The rationale is that a downswing phase of the underwriting cycle may coincide with a global recession and e.g., investment becomes unfit to compensate the underwriting losses, leaving alone dividends and other outpay. Probabilities of ruin in years of soft and hard market are considered in Section 5. Admissible risk reserve and premium controls are discussed in Section 6. Annual probability mechanisms of insurance to be used in the multiperiod model (1) are considered in Section 7.

2. Price in the years of soft and hard market

Seeking for the Lundberg-type collective model of the risk reserve process, we address first separate insurance years, or consecutive segments in the chained multiperiod model (1). In the sequel, the risk size in the portfolio is assumed stationary, i.e. the distribution of the claim amounts is the same as of a random variable Y independent on time. More light upon the economical meaning of the term "price" used below, which is rater price factor, or price rate, will be shed in Section 4, where the Lundberg-type collective model is introduced.

Definition 2.1 (Market price factor). The insurance price P^M prevailing in the market is called market price, or market price factor.

The year of soft market occurs for a particular insurer when the market price factor is below the averaged losses $\mathsf{E}Y$. Hard market for a particular insurer occurs otherwise.

Definition 2.2 (Years of soft and hard market). The insurance year is called year of soft market (for a particular insurer), if $\mathsf{E}Y > P^M$. The insurance year is called year of hard market (for a particular insurer), if $\mathsf{E}Y < P^M$.

2.1. Insurer's price control and price deficiency. Denote the insurer's price factor by P. Given P^M, selection of P for a particular portfolio means a price control. It may be done in a number of different ways and entails different consequences.

Definition 2.3 (Insurer's price control). Assume that $P = P^M$. Then the insurer applies *maintaining market share* control. Assume that $P = \mathsf{E}Y$. Then the insurer applies *conserving capital* control. The insurer applies *mixed* control, if $P^M < P < \mathsf{E}Y$, as $P^M < \mathsf{E}Y$ (soft market), and $\mathsf{E}Y < P < P^M$, as $\mathsf{E}Y < P^M$ (hard market).

Without lack of generality[1], the set $\mathcal{P}$ of price controls introduced above may be written as

$$P_\gamma = \gamma P^M + (1-\gamma)\mathsf{E}Y, \quad \gamma \in [0,1], \tag{2}$$

with $P_1 = P^M$ and $P_0 = \mathsf{E}Y$.

Remark 2.1. When $P^M < \mathsf{E}Y$ (soft market), the set $\mathcal{P} = \{P_\gamma, \gamma \in [0,1]\}$ is such that $P_{\gamma_1} > P_{\gamma_2}$ for $0 \leqslant \gamma_1 < \gamma_2 \leqslant 1$. Contrariwise, when $\mathsf{E}Y < P^M$ (hard market), the set $\mathcal{P} = \{P_\gamma, \gamma \in [0,1]\}$ is such that $P_{\gamma_1} < P_{\gamma_2}$ for $0 \leqslant \gamma_1 < \gamma_2 \leqslant 1$. When $P^M = \mathsf{E}Y$ (neutral market) the set $\mathcal{P}$ consists of a unique price.

[1]In the case of soft market (i.e., $\mathsf{E}Y > P^M$) prices P below P^M cause excessive danger of ruin, while prices P above $\mathsf{E}Y$ yield excessively high rate of elimination of portfolio. Both are claimed unreasonable. The similar arguments are true in the case of hard market.

Definition 2.4 (Price deficiency). For $\gamma \in [0,1]$ and for the price $P_\gamma \in \mathcal{P}$, the value

$$d_\gamma = P_\gamma - P^M = (1-\gamma)(\mathsf{E}Y - P^M)$$

is called price deficiency of P_γ with respect to the market price factor P^M.

Lemma 2.1. *In the year of soft market (i.e., as $\mathsf{E}Y > P^M$) price P_γ is monotonically decreasing, as γ increases, with $P_0 = \mathsf{E}Y$ and $P_1 = P^M$. Price deficiency $d_\gamma = (1-\gamma)(\mathsf{E}Y - P^M) \geqslant 0$ is monotonically decreasing, as γ increases, with $d_0 = \mathsf{E}Y - P^M > 0$ and $d_1 = 0$.*

Remark 2.2. Straightforwardly from Definition 2.4,

$$d_\gamma \begin{cases} \in [0, \mathsf{E}Y - P^M] & \text{in the case of soft market, i.e. } P^M < \mathsf{E}Y, \\ = 0 & \text{in the case of neutral market, i.e. } P^M = \mathsf{E}Y, \\ \in [\mathsf{E}Y - P^M, 0] & \text{in the case of hard market, i.e. } P^M > \mathsf{E}Y. \end{cases}$$

One may remark else that application of the maintaining market share control $P_1 = P^M$, i.e. assuming $\gamma = 1$ in Eq. (2), yields zero deficiency whichever the insurance year may be. When $\gamma \in [0,1)$, deficiency is positive in the year of soft market and negative in the year of hard market. In the latter case $e_\gamma = -d_\gamma$ may be called price excess. In the year of neutral market, when $\mathcal{P}$ consists of a unique point, deficiency is always zero.

3. Portfolio size in the years of soft and hard market

Price deficiency influences the behavior of insureds and therefore the size of portfolio. Having deficiency positive, the insurer's price exceeds the market price, which rises the insureds outflow to other companies. That outflow is in direct proportion to deficiency, being as more intensive, as larger the deficiency is. Having deficiency negative, the insurer's price is lower than the market price, which stimulates the inflow of insureds and grows of the portfolio.

Introduce a family of portfolio size functions depending on the market price and on the insurer's price control.

Definition 3.1 (Price deficiency and portfolio size). For $\gamma \in [0,1]$ and for the prices $P_\gamma \in \mathcal{P}$ with deficiency $d_\gamma = P_\gamma - P^M$, introduce the family

$$\mathcal{L} = \{\lambda_{d_\gamma}(s),\ 0 \leqslant s \leqslant t\} \tag{3}$$

of continuous non-negative functions of time, called *portfolio size* functions. Assume that $\lambda_{d_\gamma}(0) = \lambda$. The value λ is referred to as the *initial portfolio size*. In the case of $d_\gamma = 0$ (neutral market or maintaining market share control, $P_1 = P^M$) set $\lambda_{d_\gamma}(s) \equiv \lambda$, $0 \leqslant s \leqslant t$. When $d_\gamma > 0$ (soft market and $\gamma \in [0,1)$), the portfolio size functions $\lambda_{d_\gamma}(s)$ must be monotonically decreasing[2] in s and $\lambda_{d_{\gamma_1}}(s) < \lambda_{d_{\gamma_2}}(s)$ for all $0 \leqslant s \leqslant t$, as $d_{\gamma_1} > d_{\gamma_2}$. When $d_\gamma < 0$ (hard market and $\gamma \in [0,1)$), the portfolio size functions $\lambda_{d_\gamma}(s)$ must be monotonically increasing in s and[3] $\lambda_{d_{\gamma_1}}(s) < \lambda_{d_{\gamma_2}}(s)$ for all $0 \leqslant s \leqslant t$, as $d_{\gamma_1} > d_{\gamma_2}$ (i.e., $e_{\gamma_1} < e_{\gamma_2}$).

Remark 3.1. Is noteworthy that the deficiency corresponding to the maintaining market share control $P_1 = P^M$ is always zero, which yields constant portfolio size functions $\lambda_{d_1}(s) \equiv \lambda$, $0 \leqslant s \leqslant t$, whichever the insurance year may be. That agrees with that control's name.

[2]In fact, they may be non-increasing, but we exclude for simplicity the segments of constant behavior as degenerate.

[3]When $d_\gamma < 0$, the excess is positive, $e_\gamma = -d_\gamma > 0$. Evidently, the inequality $e_{\gamma_1} < e_{\gamma_2}$ is equivalent to $d_{\gamma_1} > d_{\gamma_2}$.

Introduce a sub-family of $\mathcal{L}$, with the year-end sizes bounded from below.

DEFINITION 3.2 (Portfolio size with lower bounds). For $\gamma \in [0,1]$, $L \in [0,\lambda]$, where λ is the initial portfolio size, and for $P_\gamma \in \mathcal{P}$ with deficiency d_γ, set

$$\mathcal{L}_L = \{\lambda_{d_\gamma}(s),\ 0 \leqslant s \leqslant t : \lambda_{d_\gamma}(t) \geqslant L\} \subseteq \mathcal{L}.$$

It is easily seen that $\mathcal{L}_L$ may differ from $\mathcal{L}$ only when $d_\gamma > 0$ (soft market and $\gamma \in [0,1)$). Evidently, $\mathcal{L}_\lambda$ consists of the unique constant function λ, and $\mathcal{L}_0 = \mathcal{L}$.

For $0 \leqslant s \leqslant t$, $\gamma \in [0,1]$, a few heuristic examples of $\mathcal{L}$ are: exponential $\mathcal{L}_{\exp}$, such that

$$\lambda_{d_\gamma}(s) = \lambda \exp\{-d_\gamma s\}, \tag{4}$$

linear $\mathcal{L}_{\text{lin}}$, such that

$$\lambda_{d_\gamma}(s) = \sup\big\{\lambda - d_\gamma s, 0\big\}, \tag{5}$$

power $\mathcal{L}_{\text{pow}}$, such that

$$\lambda_{d_\gamma}(s) = \sup\big\{\lambda - \operatorname{sign}(d_\gamma) s^{p(d_\gamma)}, 0\big\}, \tag{6}$$

where $p(0) = 0$, logarithmic $\mathcal{L}_{\log}$, such that

$$\lambda_{d_\gamma}(s) = \sup\big\{\lambda - d_\gamma \ln(1+s), 0\big\}. \tag{7}$$

It is noteworthy that selecting $\mathcal{L}$, wise is to address to practice. For example, one may be based on the following remark (quoted from [**29**], p. 39): "Surveys of policyholders have consistently demonstrated some reluctance to switch insurers. In a survey of 2462 policyholders by Cummins et al. [**7**], 54% of respondents confessed never to have shopped around for auto insurance prices. To the question "Which is the most important factor in your decision to buy insurance?", 40% responded the company, 29% the agent, and only 27% the premium. A similar survey of 2004 Germans (see [**27**]) indicated that, despite the fact that 67% of those responding knew that considerable price differences exist between automobile insurers, only 35% chose their carrier on the basis of their favorable premium. Therefore, we will assume that, given the opportunity to switch for a reduced premium, one-third of the policyholders will do so".

The concept of the set $\mathcal{L}$ of portfolio size functions has to be further developed. For example, it may be sensible to allow dependence of the portfolio size functions on the initial risk reserve[4].

4. Annual risk reserve process

Assume that fixed are the families $\mathcal{P}$ and $\mathcal{L}$ (see Eq. (2), (3)) of the price controls and the portfolio size functions.

DEFINITION 4.1 (Claim number process). For $P_\gamma \in \mathcal{P}$ with deficiency d_γ and for the corresponding portfolio size function $\lambda_{d_\gamma} \in \mathcal{L}$, the *claim number* process[5] is the nonhomogeneous Poisson process $\nu_\gamma(s)$, $0 \leqslant s \leqslant t$, with the yield function $\Lambda_{d_\gamma}(s) = \int_0^s \lambda_{d_\gamma}(z)dz$, $0 \leqslant s \leqslant t$.

In particular, $\mathsf{E}\nu_\gamma(s) = \Lambda_{d_\gamma}(s)$, $0 \leqslant s \leqslant t$. In the year of neutral market, $d_\gamma = 0$ and $\Lambda_{d_\gamma}(s) = \lambda s$, $0 \leqslant s \leqslant t$, which means that the growth of the mean number of claims is stable, with the constant rate λ. In the year of soft market, $\Lambda_{d_\gamma}(s) = \int_0^s \lambda_{d_\gamma}(z)dz \leqslant \lambda s$ and the growth of the mean number of claims is as slower, as larger is the time, or,

[4]It is arguable that the outflow of insureds becomes more intensive from e.g., a smaller company, for not to mention such an abstract term as the initial risk reserve. That may be checked by means of a survey of policyholders.

[5]Generated by the portfolio of variable size $\lambda_{d_\gamma}(s)$, $0 \leqslant s \leqslant t$.

which is equivalent, as smaller is the portfolio size. In the year of hard market, $\Lambda_{d_\gamma}(s) = \int_0^s \lambda_{d_\gamma}(z)dz \geqslant \lambda s$ and the effect is opposite.

LEMMA 4.1. *In the year of soft market (i.e., as* $\mathsf{E}Y > P^M$*) the year-end portfolio size* $\lambda_{d_\gamma}(t)$ *and the yield function*

$$\Lambda_{d_\gamma}(t) = \int_0^t \lambda_{d_\gamma}(z)dz$$

are monotonically increasing, as γ *increases.*

DEFINITION 4.2 (Claim outcome process). Assume that Y_i, $i = 1, 2, \dots$, are i.i.d. and independent on the claim number process $\nu_\gamma(s)$, $0 \leqslant s \leqslant t$. The *claim outcome* process associated with the portfolio size function $\lambda_{d_\gamma} \in \mathfrak{L}$ is the compound nonhomogeneous Poisson process

$$Y_\gamma(s) = \sum_{i=1}^{\nu_\gamma(s)} Y_i, \tag{8}$$

as $\nu_\gamma(s) > 0$, or zero, as $\nu_\gamma(s) = 0$, $0 \leqslant s \leqslant t$.

DEFINITION 4.3 (Premium income process). The *premium income* process associated with the portfolio size function $\lambda_{d_\gamma} \in \mathfrak{L}$ and with the premium factor P_γ is the non-random process

$$P_\gamma \Lambda_{d_\gamma}(s) = P_\gamma \int_0^s \lambda_{d_\gamma}(z)dz, \quad 0 \leqslant s \leqslant t. \tag{9}$$

As will be seen later, Definitions 4.2 and 4.3 are essential. We assume premium income process harmonized in a particular way with the claim outcome process by means of $\lambda_{d_\gamma} \in \mathfrak{L}$. Intuition suggests that it is sensible because the premium income at each time s, $0 \leqslant s \leqslant t$, has to be in direct proportion to the variable portfolio size and depends on the factor P_γ allowing for the size of risk. Emphasize it now that P_γ is selected independent on time since the size of i.i.d. risks Y_i, $i = 1, 2, \dots$, generated by that portfolio is assumed independent on time.

DEFINITION 4.4 (Risk reserve process). The *risk reserve* process generated by the premium income process (9) and claim outcome processes (8) is the random process

$$R_{u,\gamma}(s) = u + P_\gamma \Lambda_{d_\gamma}(s) - \sum_{i=1}^{\nu_\gamma(s)} Y_i, \tag{10}$$

as $\nu_\gamma(s) > 0$, or $u + P_\gamma \Lambda_{d_\gamma}(s)$, as $\nu_\gamma(s) = 0$, $0 \leqslant s \leqslant t$. The value $u > 0$ is called the *initial risk reserve*.

LEMMA 4.2. *For the claim number process* $\nu_\gamma(s)$*,* $0 \leqslant s \leqslant t$*, one has*

$$\nu_\gamma(s) = N_\lambda(\Lambda_{d_\gamma}(s)/\lambda), \quad 0 \leqslant s \leqslant t,$$

where $N_\lambda(s)$*,* $0 \leqslant s \leqslant t$*, is the homogeneous Poisson process with intensity* λ*. Moreover, for the risk reserve process* (10)*,*

$$R_{u,\gamma}(s) = u + P_\gamma \Lambda_{d_\gamma}(s) - \sum_{i=1}^{\nu_\gamma(s)} Y_i = u + [P_\gamma\lambda](\Lambda_{d_\gamma}(s)/\lambda) - \sum_{i=1}^{N_\lambda(\Lambda_{d_\gamma}(s)/\lambda)} Y_i.$$

PROOF. See e.g., [**5**], Theorem 1 on p. 38. □

LEMMA 4.3. *Introduce*

$$\hat{R}_{u,\gamma}(s) = u + [P_\gamma \lambda]s - \sum_{i=1}^{N_\lambda(s)} Y_i, \quad 0 \leqslant s \leqslant \Lambda_{d_\gamma}(t)/\lambda. \tag{11}$$

For $\tau(s) = \Lambda_{d_\gamma}(s)/\lambda$, $0 \leqslant s \leqslant t$,

$$R_{u,\gamma}(s) = \hat{R}_{u,\gamma}(\tau(s)), \quad 0 \leqslant s \leqslant t. \tag{12}$$

PROOF. The proof is standard (see [**5**], p. 38–39 and Section 2.2.3, or [**16**], Section 2.1 on p. 33, or [**2**], Remark 1.6 on p. 60). The time $\tau(s) = \Lambda_{d_\gamma}(s)/\lambda$, $0 \leqslant s \leqslant t$, is known under the name of *operational time*. Plainly, the passage of that time is no longer measured in calendar units, but in expected number of claims. □

Direct corollary of Eq. (12) is

$$\inf_{0 \leqslant s \leqslant t} R_{u,\gamma}(s) = \inf_{0 \leqslant s \leqslant \Lambda_{d_\gamma}(t)/\lambda} \hat{R}_{u,\gamma}(s). \tag{13}$$

5. Annual probabilities of ruin

DEFINITION 5.1 (Probability of ruin). The probability

$$\mathsf{P}\{\inf_{0 \leqslant s \leqslant t} R_{u,\gamma}(s) < 0\}$$

is called *annual probability of ruin*, or *probability of ruin* within time t.

THEOREM 5.1. *In the year of soft market* (*i.e., as* $\mathsf{E}Y > P^M$) *the probability*

$$\mathsf{P}\{\inf_{0 \leqslant s \leqslant t} R_{u,\gamma}(s) < 0\}$$

is monotonically increasing, as γ *increases.*

PROOF. Bearing in mind Eq. (2) and (13), one has

$$\begin{aligned}\mathsf{P}\{\inf_{0 \leqslant s \leqslant t} R_{u,\gamma}(s) < 0\} &= \mathsf{P}\Big\{\inf_{0 \leqslant s \leqslant \Lambda_{d_\gamma}(t)/\lambda} \hat{R}_{u,\gamma}(s) < 0\Big\} \\ &= \mathsf{P}\Big\{\inf_{0 \leqslant s \leqslant \Lambda_{d_\gamma}(t)/\lambda} \Big(u + \underbrace{[\mathsf{E}Y - \gamma(\overbrace{\mathsf{E}Y - P^M}^{>0})]}_{c_\gamma} \lambda s - \sum_{i=1}^{N_\lambda(s)} Y_i\Big) < 0\Big\}.\end{aligned}$$

Evidently, in the year of soft market c_γ is monotone decreasing, as γ increases, from $c_0 = \mathsf{E}Y$ to $c_1 = P^M$, with $c_0 > c_1$. By Lemma 4.1, $\Lambda_{d_\gamma}(t)$ is monotone increasing, as γ increases. Both factors contribute to a monotone growth of $\mathsf{P}\{\inf_{0 \leqslant s \leqslant t} R_{u,\gamma}(s) < 0\}$, as γ increases, which completes the proof. □

REMARK 5.1. The assertion of Theorem 5.1 is not evident, and may be incorrect, as the simultaneous decrease of cumulative premiums and of compound claims is not balanced: the former makes the probability of ruin larger, while the latter acts reversely, and without assumptions like in Definitions 4.2 and 4.3 either may be dominating.

In the case of exponential claim size, an explicit expression for the annual probability of ruin is available.

THEOREM 5.2. *Assume that* Y_i, $i = 1, 2, \ldots$, *are i.i.d. exponential with intensity* μ *and denote by* $I_n(z)$ *the modified Bessel function of* n*th order,* z *real and* $n = 0, 1, 2, \ldots$. *In that*[6] *model*

$$\mathsf{P}\{\inf_{0 \leqslant s \leqslant t} R_{u,\gamma}(s) < 0\} = \boldsymbol{\psi}_{\Lambda_{d_\gamma}(t)/\lambda}(u, [P_\gamma \lambda]), \tag{14}$$

[6] Often called “classical”.

where

$$\boldsymbol{\psi}_{\Lambda(t)/\lambda}(u,[P\lambda]) = e^{-u\mu}\sum_{n\geqslant 0}\frac{(u\mu)^n}{n\,!}\,(P\mu)^{-(n+1)/2} \times \int_0^{\Lambda(t)} \frac{n+1}{x} e^{-(1+P\mu)x} I_{n+1}(2x\sqrt{P\mu})\,dx. \tag{15}$$

The alternative expression for $\boldsymbol{\psi}_{\Lambda(t)/\lambda}(u,[P\lambda])$ *is*

$$\boldsymbol{\psi}_{\Lambda(t)/\lambda}(u,[P\lambda]) = \boldsymbol{\psi}_{\Lambda(\infty)/\lambda}(u,[P\lambda]) - \frac{1}{\pi}\int_0^{\pi} f_t(x,u)\,dx, \tag{16}$$

where

$$\boldsymbol{\psi}_{\Lambda(\infty)/\lambda}(u,[P\lambda]) = \begin{cases}(1/P\mu)\exp\{-u\mu(1-1/P\mu)\}, & P\mu>1,\\ 1, & P\mu\leqslant 1\end{cases}$$

and

$$\begin{aligned} f_t(x,u) &= (P\mu)^{-1}(1+(P\mu)^{-1}-2(P\mu)^{-1/2}\cos x)^{-1}\\ &\times \exp\Big\{u\mu\Big((P\mu)^{-1/2}\cos x - 1\Big) - \Lambda(t)P\mu\Big(1+(P\mu)^{-1}-2(P\mu)^{-1/2}\cos x\Big)\Big\}\\ &\times\Big[\cos\Big(u\mu(P\mu)^{-1/2}\sin x\Big) - \cos\Big(u\mu(P\mu)^{-1/2}\sin x + 2x\Big)\Big]. \end{aligned}$$

PROOF. The proof is straightforward from Eq. (13) and Corollary 2.1 in [**22**]. □

REMARK 5.2. To prove Theorem 5.1 in the case of exponential claim size, one may depart from Eq. (15) and check straightforwardly that in the year of soft market

$$\begin{aligned} \frac{\partial}{\partial\gamma}\mathsf{P}\{\inf_{0\leqslant s\leqslant t} R_{u,\gamma}(s)<0\} &= e^{-u\mu}(\mathsf{E}Y - P^M)\sum_{n\geqslant 0}\frac{(u\mu)^n}{n\,!}\\ &\times\int_0^{\Lambda_{d_\gamma}(t)}\Big(\mu x\Big\{\hat{v}_{n+1}(x,P_\gamma\mu) - \frac{n+1}{n+2}\,\hat{v}_{n+2}(x,P_\gamma\mu)\Big\}\Big)dx\\ &+ e^{-u\mu}\cdot\frac{\partial}{\partial\gamma}\Lambda_{d_\gamma}(t)\cdot\sum_{n\geqslant 0}\frac{(u\mu)^n}{n\,!}\hat{v}_{n+1}(\Lambda_{d_\gamma}(t),P_\gamma\mu)\geqslant 0, \end{aligned}$$

where

$$\hat{v}_{n+1}(x,P_\gamma\mu) = (P_\gamma\mu)^{-\frac{n+1}{2}}\cdot\frac{n+1}{x}e^{-(1+P_\gamma\mu)x}I_{n+1}(2x\sqrt{P_\gamma\mu}).$$

6. Admissible risk reserve and premium controls

In the model of Section 5 the controls of two different kinds are feasible: the initial risk reserve control and the premium control. In the year of soft market, admissible are those controls which do not compel the annual probability of ruin be larger than a prescribed value $\alpha\in(0,1)$, and the year-end portfolio size be less than a prescribed lower limit L.

6.1. Admissible risk reserve controls. That kind of control requires provisions made e.g., during the upswing phase of the underwriting cycle. It is based on the easy observation that the probability $\mathsf{P}\{\inf_{0\leqslant s\leqslant t} R_{u,\gamma}(s)<0\}$ is monotone decreasing, as the initial risk reserve u increases.

DEFINITION 6.1 (Least allowed initial risk reserve). For sufficiently small $\alpha \in (0,1)$, for the price factor $P_\gamma \in \mathcal{P}$ and for the portfolio size function $\lambda_{d_\gamma} \in \mathcal{L}$, call

$$u_{t,\gamma|\mathcal{L}}(\alpha) = \inf\{u > 0 : \mathsf{P}\{\inf_{0 \leqslant s \leqslant t} R_{u,\gamma}(s) < 0\} = \alpha\}$$

least allowed value of the initial risk reserve[7] satisfying the α-solvency condition.

For $\alpha \in (0,1)$ introduce $c_\alpha = \Phi^{-1}_{\{0,1\}}(1-\alpha/2)$, where $\Phi_{\{0,1\}}(\cdot)$ is the standard normal distribution function.

THEOREM 6.1. *In the framework of Theorem 5.2, the least allowed value of the initial risk reserve satisfying the α-solvency condition is*

$$u_{t,\gamma|\mathcal{L}}(\alpha) = (\mathsf{E}Y - P_\gamma)\Lambda_{d_\gamma}(t)(1+\overline{o}(1)), \quad t \to \infty,$$

as $\gamma > 1$, and

$$u_{t,1|\mathcal{L}}(\alpha) = \sqrt{2\lambda t}\ \mathsf{E}Y c_\alpha(1+\overline{o}(1)), \quad t \to \infty,$$

as $\gamma = 1$ (i.e., for the conserving capital control $P_1 = \mathsf{E}Y$).

PROOF. The proof is similar to the proof of Theorem 4.4 in [**24**] and Theorem 4.1 in [**23**]. □

6.2. Admissible premium controls. In the year of soft market[8], assuming of any premium control from $\mathcal{P}$ is an attempt of making the best of a bad bargain since each price control is attended with inevitable outflow of the insureds according to a rate from $\mathcal{L}$, and with a guaranteed deterioration (except, perhaps, for $P_0 = \mathsf{E}Y$) of the solvency position. On the other hand, that kind of control does not require provisions.

6.2.1. *Solvency point of view.* From the solvency point of view, admissible premium control is based on the following.

THEOREM 6.2. *For sufficiently small $\alpha \in (0,1)$, for the initial risk reserve u and for the family $\mathcal{L}$, in the year of soft market allowed are the price controls $P_\gamma \in \mathcal{P}$, $\gamma \in [0, \gamma_{t,u|\mathcal{L}}(\alpha)]$, where $\gamma_{t,u|\mathcal{L}}(\alpha)$ is the unique solution of the equation*

$$\mathsf{P}\{\inf_{0 \leqslant s \leqslant t} R_{u,\gamma}(s) < 0\} = \alpha, \tag{17}$$

as $\mathsf{P}\{\inf_{0\leqslant s\leqslant t} R_{u,1}(s) < 0\} \geqslant \alpha$, and $\gamma_{t,u|\mathcal{L}}(\alpha) = 1$, as $\mathsf{P}\{\inf_{0\leqslant s\leqslant t} R_{u,1}(s) < 0\} < \alpha$.

PROOF. This result follows directly from Theorem 5.1 which claims monotone increasing of $\mathsf{P}\{\inf_{0\leqslant s\leqslant t} R_{u,\gamma}(s) < 0\}$, as γ increases, in the year of soft market. It shows that allowed are the price controls from a right neighborhood of the conserving capital control $P_0 = \mathsf{E}Y$. □

In the classical case the numerical solution $\gamma_{t,u|\mathcal{L}}(\alpha)$ of Eq. (17) is easy to get by means of Eq. (14) and (16) which yield an explicit expression for $\mathsf{P}\{\inf_{0\leqslant s\leqslant t} R_{u,\gamma}(s) < 0\}$.

We address the analytical approach. The key point is to devise an explicit expression, or a manageable approximation to the left hand side of Eq. (17).

To formulate the first result in that direction, introduce a shorter notation: put $\gamma_{t,\alpha}$ for $\gamma_{t,u|\mathcal{L}}(\alpha)$, set

$$\mathsf{P}\{\inf_{0\leqslant s\leqslant t} R_{u,\gamma}(s) < 0\} = \boldsymbol{\psi}_t(\gamma)$$

[7]Evidently, $\mathsf{P}\{\inf_{0\leqslant s\leqslant t} R_{u,\gamma}(s) < 0\} < \alpha$ for $u > u_{t,\gamma|\mathcal{L}}(\alpha)$.

[8]The counterparts of the results below in the case of hard market are omitted for brevity, but obtained by the author.

and note that in the year of soft market $\mathsf{P}\{\inf_{0\leqslant s\leqslant\infty} R_{u,\gamma}(s) < 0\} = \boldsymbol{\psi}_{+\infty}(\gamma) = 1$. Introduce the probability of ultimate ruin after time t,

$$\phi_t(\gamma) = \boldsymbol{\psi}_{+\infty}(\gamma) - \boldsymbol{\psi}_t(\gamma) = 1 - \boldsymbol{\psi}_t(\gamma),$$

and note that $\phi_t(\gamma_{t,\alpha}) = 1 - \boldsymbol{\psi}_t(\gamma_{t,\alpha}) = 1 - \alpha$, which yields

$$\gamma_{t,\alpha} = \phi_t^{-1}(1-\alpha).$$

THEOREM 6.3. *For* $\tau_\gamma = -\gamma(\mathsf{E}Y - P^M)/\mathsf{E}Y$, $\gamma \in (0,1]$, *set* $a_\gamma = (1-\sqrt{1+\tau_\gamma})^2$ *and* $b_\gamma = 1/\sqrt{1+\tau_\gamma}$. *In the framework of Theorem 5.2, one has* $\tau_\gamma < 0$, *and*

$$\phi_t(\gamma) = \frac{b_\gamma^{3/2}(b_\tau u\mu + 1)}{2\sqrt{\pi}a_\gamma(\Lambda_{d_\gamma}(t))^{3/2}}\, e^{-u\mu(1-b_\gamma)} e^{-a_\gamma\Lambda_{d_\gamma}(t)} \exp\Big\{-\frac{b_\gamma^3(u\mu)^2}{4\Lambda_{d_\gamma}(t)}\Big\}\Big\{1 + \underline{O}(\Lambda_{d_\gamma}^{-1/2}(t))\Big\}$$

for $u \leqslant \underline{O}(\Lambda_{d_\gamma}^{1/2}(t))$, *as* $t \to \infty$.

PROOF. The proof applies explicit expression (15) in Theorem 5.2 and the expansions technique introduced in Section 3 of [**23**]. That result is a counterpart of one in [**30**]. □

The second result, suitable for $u \geqslant \underline{O}(\Lambda_{d_\gamma}^{1/2}(t))$, as $t \to \infty$, is the following Normal approximation.

THEOREM 6.4. *For* $\tau_\gamma = -\gamma(\mathsf{E}Y - P^M)/\mathsf{E}Y$, $\gamma \in (0,1]$, *assume that* $\tau_\gamma < 0$. *Then*

$$\sup_{t\in\mathsf{R}^+} \big|\boldsymbol{\psi}_t(\gamma) - \Phi_{\{0,1\}}((\Lambda_{d_\gamma}(t) - M_{\tau_\gamma}u\mu)/(S_{\tau_\gamma}(u\mu)^{1/2}))\big| = \underline{O}(u^{-1/2}), \text{ as } u \to \infty,$$

where $M_{\tau_\gamma} = -1/\tau_\gamma$, $S_{\tau_\gamma}^2 = -2/\tau_\gamma^3$.

PROOF. The proof is easy and follows either from Theorem 5 (I) in [**19**], or from Section 3.1 in [**22**]. □

Asymptotic expansions in Theorem 6.4 may be further developed following [**19**], and large deviations, as $u \gg \underline{O}(\Lambda_{d_\gamma}(t))$, as $t \to \infty$, may be obtained following [**20**].

6.2.2. *Portfolio size point of view.* From the portfolio size point of view, admissible premium control is based on the following.

THEOREM 6.5. *For sufficiently small* $\alpha \in (0,1)$, *for the initial risk reserve* u *and for the family* $\mathcal{L}$, *in the year of soft market allowed are the price controls* $P_\gamma \in \mathcal{P}$, $\gamma \in [\gamma_L, 1]$, *where*

$$\gamma_L = \inf\{\gamma \in [0,1] : \lambda_{d_\gamma}(t) = L\} > 0,$$

as $\lambda_{d_0}(t) < L$, *and* $\gamma_L = 0$, *as* $\lambda_{d_0}(t) \geqslant L$.

PROOF. This result follows directly from Lemma 4.1 which claims monotone increasing of the year-end portfolio size $\lambda_{d_\gamma}(t)$, as γ increases, in the year of soft market. It shows that allowed are the price controls from a right neighborhood of the maintaining market share control $P_1 = P^M$. □

COROLLARY 6.1. Theorems 6.2 and 6.5 yield the set of the annual price controls allowed from both solvency and portfolio size points of view. This set is

$$P_\gamma \in \mathcal{P}, \quad \gamma \in [0, \gamma_{t,u|\mathcal{L}}(\alpha)] \cap [\gamma_L, 1] = [\gamma_L, \gamma_{t,u|\mathcal{L}}(\alpha)].$$

6.3. A strategy beating the downswing phase of the cycle. Further preference among the admissible controls is unspecified until more criteria are introduced. It may be noteworthy that the set of admissible controls is dependent on the initial risk reserve u. Typically, the latter is set equal to the risk reserve at the previous year-end. In that sense the consecutive annual controls, depending on the previous year financial result, give rise to an *adaptive* control strategy.

For the family $\mathfrak{L}$ and for a sequence $u, w_1, \dots, w_{k-1}$ of the initial risk reserve values[9], as the $(i-1)$st year-end risk reserve is assumed equal to the initial risk reserve in ith year $(i = 2, \dots, k)$, the adaptive control strategy beating the downswing phase of the insurance cycle with the period k, generated by the market prices $P_1^M > \dots > P_k^M > 0$, all below the average risk $\mathsf{E}Y$, is

$$\begin{aligned} &P_1(u) = P_\gamma, \quad \gamma \in [\gamma_L, \gamma_{t,u|\mathfrak{L}}(\alpha_1)], && \text{if} \quad [\gamma_L, \gamma_{t,u|\mathfrak{L}}(\alpha_1)] \neq \varnothing, \\ &P_2(w_1) = P_\gamma, \quad \gamma \in [\gamma_L, \gamma_{t,w_1|\mathfrak{L}}(\alpha_2)], && \text{if} \quad [\gamma_L, \gamma_{t,w_1|\mathfrak{L}}(\alpha_2)] \neq \varnothing, \\ &\dots\dots \\ &P_k(w_{k-1}) = P_\gamma, \quad \gamma \in [\gamma_L, \gamma_{t,w_{k-1}|\mathfrak{L}}(\alpha_k)], && \text{if} \quad [\gamma_L, \gamma_{t,w_{k-1}|\mathfrak{L}}(\alpha_k)] \neq \varnothing. \end{aligned} \tag{18}$$

Recall (see (1)) that $\alpha_1, \dots, \alpha_k$ are the allowed levels or ruin within the downswing phase of the insurance cycle[10].

Plainly, for $\alpha > 0$ and for $u \geqslant w_1 \geqslant \dots \geqslant w_{k-1}$, which is the typical case in the downswing phase, one has $\gamma_{t,u|\mathfrak{L}}(\alpha) \geqslant \gamma_{t,w_1|\mathfrak{L}}(\alpha) \geqslant \dots \geqslant \gamma_{t,w_{k-1}|\mathfrak{L}}(\alpha)$. Rigorous probability model for the trajectories $(u, w_1, \dots, w_{k-1})$, formalizing the diagram (1), requires a formal definition of the annual probability mechanisms of insurance.

7. Annual probability mechanism of insurance

In this section we consider the annual probability mechanisms of insurance π_k fit to the multiperiod model (1) with the adaptive[11] control strategy (18). In that model no risk reserve control is applied: each year the initial risk reserves will be set equal to the risk reserve at the previous year-end.

Consider the risk reserve process (10),

$$R_{u,\gamma}(s) = u + P_\gamma \, \Lambda_{d_\gamma}(s) - \sum_{i=1}^{\nu_\gamma(s)} Y_i,$$

where $0 \leqslant s \leqslant t$. For $u \in \mathsf{R}^+$ and Borel set A introduce the kernel

$$\begin{aligned} \pi_t(A, \text{ruin} \mid u, P_\gamma) &= \mathsf{P}\big\{R_{u,\gamma}(t) \in A, \inf_{0<s\leqslant t} R_{u,\gamma}(s) < 0\big\}, \\ \pi_t(A, \text{no ruin} \mid u, P_\gamma) &= \mathsf{P}\big\{R_{u,\gamma}(t) \in A, \inf_{0<s\leqslant t} R_{u,\gamma}(s) > 0\big\} \\ &= \mathsf{P}\{R_{u,\gamma}(t) \in A\} - \pi_t(A, \text{ruin} \mid u, P_\gamma) \end{aligned}$$

and note that

$$0 \leqslant \pi_t(A, \text{ruin} \mid u, P_\gamma) \leqslant \mathsf{P}\{\inf_{0\leqslant s\leqslant t} R_{u,\gamma}(s) < 0\}$$

and

$$0 \leqslant \pi_t(A, \text{no ruin} \mid u, P_\gamma) \leqslant \mathsf{P}\{R_{u,\gamma}(t) \in A\}.$$

---

[9]Bearing in mind diagram (1), more accurate is to say that $w_1, \dots, w_{k-1}$, being the year-end values of the risk reserve, are the state variables, and $u_{i-1} = w_{i-1}$, $i = 2, \dots, k$, being the initial risk reserves, are the control variables; the vector of the control variables is therefore $u, u_1, \dots, u_{k-1}$.

[10]For capital-dependent $\mathfrak{L}$ the values γ_L become capital-dependent.

[11]In that sense that the control depends on the previous year financial result.

More delicate analysis is possible when the time-transformed process (11) is Poisson–exponential. That analysis is based on the joined distribution of the time of ruin[12] $\hat{\tau} = \inf\{s > 0 : \hat{R}_{u,\gamma}(s) < 0\}$ and of the corresponding deficit at ruin $\hat{\delta}$ explicit (see, e.g., [**15**]) in the classical case, since

$$\begin{aligned}\pi_t(A, \text{ruin} \mid u, P_\gamma) &= \mathsf{P}\big\{\hat{R}_{u,\gamma}(\Lambda_{d_\gamma}(t)/\lambda) \in A, \inf_{0\leqslant s\leqslant \Lambda_{d_\gamma}(t)/\lambda} \hat{R}_{u,\gamma}(s) < 0\big\} \\ &= \int_0^\infty \int_0^{\Lambda_{d_\gamma}(t)/\lambda} \mathsf{P}\{\hat{\tau} \in ds, \hat{\delta} \in dy\} \mathsf{P}\{\hat{R}_{-y,\gamma}((\Lambda_{d_\gamma}(t)/\lambda) - s) \in A\}.\end{aligned}$$

It is shown in [**1**] that in the Poisson–Exponential model $\hat{\tau}$ is independent on $\hat{\delta}$; the latter is exponential with parameter μ. It yields

$$\begin{aligned}\pi_t(A, \text{ruin} \mid u, P_\gamma) &= \int_0^\infty \mathsf{P}\{\hat{\delta} \in dy\} \int_0^{\Lambda_{d_\gamma}(t)/\lambda} \mathsf{P}\{\hat{\tau} \in ds\} \mathsf{P}\{\hat{R}_{-y,\gamma}((\Lambda_{d_\gamma}(t)/\lambda) - s) \in A\} \\ &= \int_0^\infty \mu e^{-\mu y} dy \int_0^{\Lambda_{d_\gamma}(t)/\lambda} \mathsf{P}\{\hat{\tau} \in ds\} \mathsf{P}\{\hat{R}_{-y,\gamma}((\Lambda_{d_\gamma}(t)/\lambda) - s) \in A\}.\end{aligned}$$

The explicit expressions for $\pi_t(A, \text{ruin} \mid u, P_\gamma)$ are yielded therefore by Theorem 5.2 and by the results of the following kind.

THEOREM 7.1. *In the framework of Theorem 5.2, for real x,*

$$\mathsf{P}\{R_{u,\gamma}(t) \leqslant x\} = \begin{cases} 1, & x > u + P_\gamma \Lambda_{d_\gamma}(t), \\ 1 - e^{-\Lambda_{d_\gamma}(t)} - e^{-\Lambda_{d_\gamma}(t)} \sqrt{\mu \Lambda_{d_\gamma}(t)} & \\ \quad \times \displaystyle\int_0^{u + P_\gamma \Lambda_{d_\gamma}(t) - x} z^{-1/2} I_1\left(2\sqrt{\mu \Lambda_{d_\gamma}(t) z}\right) e^{-\mu z}\, dz, & x \leqslant u + P_\gamma \Lambda_{d_\gamma}(t), \end{cases}$$

where $I_1(\cdot)$ is the modified Bessel function of unit order.

PROOF. Proof is straightforward from Lemma 4.3 and Theorem 2.1 in [**22**]. □

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E-mail address: malinov@orc.ru, malinov@mi.ras.ru
URL: http://www.actlab.ru