SCENARIO ANALYSIS FOR A MULTI-PERIOD DIFFUSION MODEL OF RISK

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Abstract. This paper extends and develops the results in the paper [**9**]. Dealing with a simplistic diffusion multi-period model of insurance operations, it illustrates the adaptive control approach when the object of control is harmonization of the solvency and equity requirements. With regard to [**9**], the main novelty is the incomplete knowledge of the forthcoming risk which is quite often the case in insurance. Represented by a scenario of nature, it introduces new and inevitable randomness in the model and induces qualitative difference beside the case of completely known risk.

1. Introduction

In the papers [**7**]–[**9**] the insurance process is viewed as a series of successive insurance periods called years. Each year starts with a manager's adaptive, or sensitive to the financial results in the previous year, control decision. The insurance operations are represented by a probability mechanism. The manager's decision concerns tariffs, reserves and other operational characteristics of the probability mechanism of insurance. By the nature of insurance, that control decision typically remains in force throughout the whole insurance year, i.e., until development of the next year-end financial report and subsequent control intervention.

The adaptive control approach in insurance modelling is inspired by many scholars including K. Borch who claimed back in 1967 that "general formulation of the actuary's problem leads directly to the general theory of *optimal control processes* or *adaptive control processes*" and "the theory of control processes seems to be "tailor–made" for the problems which actuaries have struggled to formulate for more than a century" (see [**2**], p. 451).

The object of control set forth in [**7**]–[**9**] was harmonization of the solvency and equity requirements. Solvent controls mean that a prescribed probability of non-ruin must be guaranteed uniformly on the past-years financial results, whichever external particulars within certain limits might be. Equity requires premiums well-balanced with claims, loaded with an amount necessary to provide adequate security for the insureds, rather than benefit those who seek unearned profit. It means that the insureds ought to pay premiums which are sensibly concentrated around the long-run mean value of their losses. In that sense the customers will not be overcharged, but only in the long run (i.e., in the average throughout several insurance years), while in the separate insurance years the premiums may be above or below average. Insurers, spreading the cost of random losses among the policyholders, and over time, act as a buffer against claim fluctuations in consecutive years.

Key words and phrases. Multi-period insurance process, Diffusion annual mechanisms, Volatile scenario, Solvency, Equity, Adaptive control strategies.

Related is the problem of discrimination of the risk reserves, capital and special purpose provisions[1] (see, e.g., [**4**]). Bearing in mind the principle of equity, risk reserve must be held large enough to secure solvency, but at the expectation, called "target" or "fair" capital value. Otherwise, one would argue that it is used to cover the unexpected, and it is right. But the probability is small that the risk reserve will end up at the expectation at the end of the year. It will most probably be above or below the expectation and from time to time much above or below, the more so, the larger the manager's prediction disagrees with the actual dynamics of nature. It makes carrying the appropriate risk-based provisions, such that over many years one will still be at the expectation, an important problem of the insurance management. The present paper addresses that problem from theoretical premises of dynamic solvency provisions set for zone-adaptive annual controls.

The economic bearing of the object of control considered in [**7**]–[**9**] and in the present paper is therefore a cautious and equitable asset–liability and solvency adaptive management. Sophistication of the model may lead to additional rational priorities and to more complicated objects of control but does not injure the fundamental nature of the adaptive control concept of this paper.

It should be emphasized that the adaptive rather than optimal control is the main concern of the paper. The optimal control usually aims the single-purposed objectives like e.g., maximization of the insurer's profit[2]. Even under some restrictions on ruin and some kind of equity determined by the market, it yields quite a different mathematical game which lies outside the scope of the present paper. The optimal control is rather traditional set-up in actuarial mathematics (see, e.g., [**1**], [**11**]), but the objectives like "to find the policy which maximizes the expected total discounted dividend pay-outs until the time of bankruptcy" (see [**11**], p. 105) were severely criticized as deficient in the insurance context (see quotation from C.-O. Segerdhal on p. 392 of [**9**]).

Plain is the idea that insurance deals with such uncertainties as random claim arrival and random claim severity. Even more uncertainty immanent for the insurance business is due to the randomness called scenario of nature. Quoting Norbert Wiener (see [**12**], p. 90), it results in an uncomfortable resemblance of that — insurer's vs. nature — economic game to the Queen's croquet game in "Alice in Wonderland". Wiener emphasized that such resemblance exists in all economic games where the rules are subject to important and, put it in addition, random revisions. To be particular, recall for example that change of climate around us impacts and will increasingly impact many sectors of business and society. The most profound effects are likely to be associated with changes in rainfall and hazardous weather, but fortunately climate models are reaching a level of sophistication where they can be used to guide decision-making at the regional and local level.

It is recognized (see e.g., [**2**], p. 451) that the insurance company, being incompletely informed, needs to devise

(i) an *information system*: a system for observing the insurance process as it develops,
(ii) a *decision function*: a set of rules for translating the observations into action.

The latter means in particular that a manager's control which fine-tunes tariffs, reserves and other operational characteristics of the probability mechanism of insurance in a series of successive insurance years, called strategy and developed under deficiency of information, should be thoroughly analyzed by actuaries to make its impact on the insurer's business clearly understood.

Two commonly accepted techniques aimed to evaluate the impact of deficiency of information on the insurer's business are scenario analysis and stress testing. The former considers typical, favorable and unfavorable scenarios of nature. The latter refers to shifting the values of the individual

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[1] Many parties to the insurance business are very attentive to that problem by other reasons than equity: reserves belong legally to the policyholders while capital belongs to the shareholders; risk reserve should be invested at the risk free rate, while the capital can be invested in riskier and more rewarding assets; taxation of risk reserves and capital is different.

[2] Standing by the side of insurers, wise is to care for the insureds as good shepherd cares for his sheep. In that sense the position of those who wish to win clients' loyalty, or merely avoid they outflow, may agree with the object of control set forth in the paper. More technical discussion is deferred to Section 2.4

parameters in the model that affects critically the insurer's financial position. Largely, both apply simulation.

The present paper's purpose is to accentuate the risk theory-based, analytical approach. Regarding the general multi-period model of risk (the control-oriented reader may wish to start from formal definitions deferred to Section 3), each trajectory may be diagrammed as

$$\underbrace{\mathfrak{w}_0 \xrightarrow{\gamma_0} \mathfrak{u}_0 \xrightarrow{\pi_1} \mathfrak{w}_1}_{\text{1-st year}} \;\cdots\; \xrightarrow{\pi_{k-1}} \underbrace{\mathfrak{w}_{k-1} \xrightarrow{\gamma_{k-1}} \mathfrak{u}_{k-1} \xrightarrow{\pi_k} \mathfrak{w}_k}_{k\text{-th year}} \;\cdots . \tag{1}$$

According to this diagram[3] (for $k = 1, 2, \dots$), at the end of $(k-1)$-th year the state variable $\mathfrak{w}_{k-1}$ is observed. It describes the insurer's position at that moment and may be of a more complex structure than just a real-valued surplus. Then, obeying certain rules called scenario, the nature selects at the beginning of the k-th year a value influencing the forthcoming annual risk, while the control rule γ_{k-1} is applied by the insurer to choose the control variable $\mathfrak{u}_{k-1}$. The structural assumption that nature is acting first, before insurer, at the beginning of the incoming insurance year, and that the lag between their actions is negligible, may be easily weakened. In what follows, it is accepted for simplicity. Typically, making his control decision, insurer remains ignorant about the nature's choice. He acts bearing in mind the limitations induced by the scenario, if the latter is known, and applying the past-year data[4] $\mathfrak{w}_{k-1}$ to the control rule γ_{k-1}. Thereupon the k-th year probability mechanism of insurance unfolds; the transition function of this mechanism is denoted by π_k. It defines the insurer's position at the end of the k-th year, and the process repeats anew.

Paramount in (1) is the annual probability mechanism of insurance[5]. In [**7**], [**8**] it is generated by the Poisson–Exponential collective risk model, while in [**9**] and in the present paper it is diffusion: the annual probability mechanism of insurance is produced (see Section 3) by the claim out-pay process $V_s(M) = Ms + \sigma(M)\mathsf{W}_s$, $0 \leqslant s \leqslant t$, and the annual risk reserve process

$$R_s(u, c, M) = u + cs - V_s(M), \quad 0 \leqslant s \leqslant t, \tag{2}$$

where u is the risk reserve at the beginning of the year, called initial risk reserve or starting capital, c is the premium intensity, M is the random claim out-pay rate, $\sigma(\cdot)$ is a known function assuming positive values and $\sigma^2(M)$ is the random volatility; W_s, $0 \leqslant s \leqslant t$, is the standard Brownian motion. In (2), sensible control leverages are both the initial risk reserve and the premium intensity, so that the control variable is bivariate.

Having specified the annual mechanism of insurance, paramount is to keep track of how the information is revealed in time. Going back to the diagram (1), introduce the sequence $\{\mathsf{W}_s^{[k]}, 0 \leqslant s \leqslant t\}$, $k = 1, 2, \dots$, of *independent* Brownian motions and the sequence M_k, $k = 1, 2, \dots$, of the random claim intensities. Assume that these sequences are *independent* of each other. These two independence assumptions are sensible regularity ones. The former guarantees independence of the annual claim out-pay processes $V_s^{[k]}(M_k)$, $k = 1, 2, \dots$, provided the claim intensities are fixed, the latter reflects independence of the choice of nature from the particulars of the annual insurance process, which looks sensible, not to mention lobbying. To concatenate the annual probability mechanisms (see formalities in Section 3), we address the following simplistic scenario of nature.

Definition 1.1. By the volatile (homogeneous and with known generic risk) scenario of nature associated with the multi-period model (1) and the annual mechanisms of insurance (2) we mean the sequence of i.i.d. claim intensities M_k, $k = 1, 2, \dots$, with known generic distribution G.

[3] In Section 3 the state variables $\mathfrak{w}_k$, the control variables $\mathfrak{u}_k$ and the other components of the scheme (1) are yielded explicitly in the case of our particular interest.

[4] Or, more generally, all the past history $\mathfrak{Y}_{k-1} = (\mathfrak{u}_0, \dots, \mathfrak{u}_{k-2}, \mathfrak{w}_0, \dots, \mathfrak{w}_{k-1})$. The control based on $\mathfrak{Y}_{k-1}$ is called (see [**9**]) non-Markov, the control $\mathfrak{u}_{k-1} = \gamma_{k-1}(\mathfrak{w}_{k-1})$ is called Markov.

[5] The scheme (1) is fit to model non-homogeneous dynamics of the insurance process by means of addressing different annual probability mechanisms of insurance.

Going back to the multi-period model (1) with the annual mechanisms of insurance (2), under the volatile scenario of nature, fair is the premium rate $c = \mathsf{E}M$. By the law of large numbers, it equalizes in the long-run the many-years average value of the annual claims since, bearing in mind independence of M and $\{\mathsf{W}_s, 0 \leqslant s \leqslant t\}$,

$$\mathsf{E}V_t(M) = \mathsf{E}\big(Mt + \sigma(M)\mathsf{W}_t\big) = \mathsf{E}M \cdot t.$$

Appropriate selection of the two control leverages c and u in (2) must compensate purely random fluctuations of the stochastic process $V_s(m)$, $0 \leqslant s \leqslant t$, around a "target" or "fair" capital value to be defined later and the errors due to the difference between the unknown but actual realization m of the random variable M and the heuristic but average value $\mathsf{E}M$.

Following recommendations to supplement the analysis of a model with a clear warning of its restricted applicability (see [**5**], Chapter 1, Section 5.5, p. 154), emphasize it that we deal with simplistic diffusion annual mechanism (2) and simplistic volatile scenario, but do that with purpose. Simplistic model allows us a transparent mathematics, and yields a telling illustration of the adaptive control approach. For computer-oriented analysts extension on more general annual mechanisms is straightforward by means of numerical solution of the basic equations introduced in Section 2. For those who cares for more realistic probability background, variety of approaches is available. In particular, applying [**7**], [**8**], one may easily extend the results of the paper on the Poisson–Exponential case. Overall, the simplistic models may hint on how to attack more realistic insurance risk models, when no hope of such reward as explicit formulae exists.

The rest of the paper is arranged as follows. Section 2 develops the concepts introduced in the framework of complete information in [**7**]–[**9**]. Section 3 is devoted to rigorous definition of the multi-period diffusion model under the volatile scenario of nature and to the analysis of the equity and solvency properties of certain adaptive control strategies. Section 4 contains auxiliary results.

2. Synthesis of the annual adaptive controls

This section is devoted to the annual development of the insurance process. It is preparatory for the multi-period modelling and for the strategy design of Section 3. We denote by $\Phi(x)$ the standard normal distribution function and by $\phi(x)$ its density function. For $0 < \gamma < 1$, denote by $\kappa_\gamma = \Phi^{-1}(1-\gamma)$ the $(1-\gamma)$-quantile of $\Phi(x)$.

2.1. Annual solvency criteria. Formulate an assumption and two definitions.

ASSUMPTION 1. In the diffusion generic model (2) the random parameter M is non-degenerate, with c.d.f. G and support $\mathrm{M} \subset \mathsf{R}^+$.

In the framework of diffusion generic model (2), for $m \in \mathrm{M}$, set

$$\boldsymbol{\psi}_t(u,c,m) = \mathsf{P}\big\{\inf_{0 \leqslant s \leqslant t} R_s(u,c,M) < 0 \mid M = m\big\}, \quad t \geqslant 0. \tag{3}$$

The control variable is bivariate (u, c), in the sequel being a function of the past-year-end capital. Introduce two annual solvency criteria which modify the standard one.

DEFINITION 2.1. The adaptive control $(u(w), c(w))$, where w is the past-year-end capital, satisfies the α-level $(0 < \alpha < 1)$ conservative, or uniform, solvency criterion if

$$\sup_{w>0, m \in \mathrm{M}} \boldsymbol{\psi}_t(u(w), c(w), m) \leqslant \alpha. \tag{4}$$

DEFINITION 2.2. The adaptive control $(u(w), c(w))$, where w is the past-year-end capital, satisfies the α-level integral solvency criterion if

$$\sup_{w>0} \mathsf{P}\big\{\inf_{0 \leqslant s \leqslant t} R_s(u(w), c(w), M) < 0\big\} = \sup_{w>0} \int_{\mathrm{M}} \boldsymbol{\psi}_t(u(w), c(w), m)\, G(dm) \leqslant \alpha. \tag{5}$$

Remark 2.1. In the particular case of the bounded support $\mathrm{M} = [\mu_{\min}, \mu_{\max}]$, $0 < \mu_{\min} < \mu_{\max} < \infty$, when the possible choice of the nature is a priori known not to exceed the known value $\mu_{\max}$, the claim intensity $\mu_{\max}$ is the most unfavorable case for the insurer,

$$\sup_{w>0, m\in \mathrm{M}} \boldsymbol{\psi}_t(u(w), c(w), m) = \sup_{w>0} \boldsymbol{\psi}_t(u(w), c(w), \mu_{\max}),$$

and $(u(w), c(w))$ satisfies the α-level conservative solvency criterion if

$$\sup_{w>0} \boldsymbol{\psi}_t(u(w), c(w), \mu_{\max}) \leqslant \alpha. \tag{6}$$

Assumption 2. Assume in what follows that $\mathrm{M} = [\mu_{\min}, \infty)$, $0 < \mu_{\min} < \infty$, i.e., only the lower bound $\mu_{\min}$ of the claim intensity, or the most favorable case for the insurer, is a priori known.

The conservative solvency criterion may be called "egalitarian" with respect to all realizations of M since it treats alike "liable" (moderate) and "force-majeure" (large) values of M. It is exceedingly restrictive, while the integral solvency criterion attributes proper weights to different choices of the nature by means of c.d.f. G. Therefore, the latter is more probabilistic in nature.

Recall that μ_α $(0 < \alpha < 1)$ such that $\mathsf{P}\{M > \mu_\alpha\} = \alpha$, or $G(\mu_\alpha) = 1 - \alpha$, is called $(1-\alpha)$-quantile of c.d.f. G. Since we are not seeking for generality, introduce the following assumption which guarantees that μ_α exists and is unique.

Assumption 3. Assume that G is absolutely continuous.

The reader will easily extend the arguments to e.g., discrete G and bounded M.

Definition 2.3. The adaptive control $(u(w), c(w))$, where w is the past-year-end capital, satisfies the (α_1, α_2)-solvency criterion with $\alpha_i \in (0, 1/2)$, $i = 1, 2$, if for the $(1-\alpha_1)$-quantile μ_{α_1} of c.d.f. G

$$\sup_{w>0, m\leqslant \mu_{\alpha_1}} \boldsymbol{\psi}_t(u(w), c(w), m) \leqslant \alpha_2. \tag{7}$$

The adaptive control $(u(w), c(w))$ satisfies the (α_1, α_2)-solvency criterion sharply if

$$\boldsymbol{\psi}_t(u(w), c(w), \mu_{\alpha_1}) = \alpha_2$$

for all $w > 0$.

Theorem 2.1 (Sufficient conditions of integral solvency). *Assume that the adaptive control $(u(w), c(w))$ satisfies the (α_1, α_2)-solvency criterion. Then it satisfies the $(\alpha_1 + \alpha_2)$-level integral solvency criterion.*

Proof of Theorem 2.1. It is noteworthy that

$$\sup_{w>0, m\leqslant \mu_{\alpha_1}} \boldsymbol{\psi}_t(u(w), c(w), m) = \sup_{w>0} \boldsymbol{\psi}_t(u(w), c(w), \mu_{\alpha_1}).$$

Bearing in mind (5), the simple inequalities

$$\begin{aligned}\sup_{w>0} \mathsf{P}\Big\{\inf_{0\leqslant s\leqslant t} R_s(u(w), c(w), M) < 0\Big\} &\leqslant \sup_{w>0} \int_{m\leqslant \mu_{\alpha_1}} \boldsymbol{\psi}_t(u(w), c(w), m)\, G(dm) + \int_{m>\mu_{\alpha_1}} G(dm) \\ \leqslant \sup_{w>0, m\leqslant \mu_{\alpha_1}} \boldsymbol{\psi}_t(u(w), c(w), m) + \mathsf{P}\{M > \mu_{\alpha_1}\} &= \sup_{w>0} \boldsymbol{\psi}_t(u(w), c(w), \mu_{\alpha_1}) + \alpha_1 \leqslant \alpha_2 + \alpha_1\end{aligned}$$

yield the result. □

We will be concerned mostly about the controls which satisfy the (α_1, α_2)-solvency criterion and, consequently, the $(\alpha_1 + \alpha_2)$-level integral solvency criterion. It means that we can confine ourselves to $m \in [\mu_{\min}, \mu_{\alpha_1}]$, $\mu_{\min} > 0$, and disregard the other outcomes "of rare occurrence".

2.2. Level capital and premium intensity. Introduce two key components of the adaptive control rules. The existence and the structure of these components in the diffusion framework will be discussed in Theorems 4.4 and 4.5.

DEFINITION 2.4. For $\alpha_i \in (0, 1/2)$, $i = 1, 2$, and for the $(1 - \alpha_1)$-quantile μ_{α_1} of c.d.f. G the solution $u_{\alpha_2,t}(c, \mu_{\alpha_1})$ of the equation

$$\boldsymbol{\psi}_t(u, c, \mu_{\alpha_1}) = \alpha_2 \tag{8}$$

with respect to u is called α_2-level initial capital corresponding to the claim intensity μ_{α_1} and premium intensity c. The solution $c_{\alpha_2,t}(u, \mu_{\alpha_1})$ of Eq. (8) with respect to c is called α_2-level premium intensity corresponding to the claim intensity μ_{α_1} and initial capital u.

REMARK 2.2. By definition, $c_{\alpha_2,t}(u_{\alpha_2,t}(c, \mu_{\alpha_1}), \mu_{\alpha_1}) = c$, $u_{\alpha_2,t}(c_{\alpha_2,t}(u, \mu_{\alpha_1}), \mu_{\alpha_1}) = u$.

2.3. Rigid (non-adaptive) controls. Disregard of principles other than solvency may lead to a safe, but unsatisfactory control. Demonstrate it by means of two simple illustrative examples. For the past-year-end capital w consider $\alpha_i \in (0, 1/2)$, $i = 1, 2$, and $\mu \in [\mu_{\min}, \mu_{\alpha_1}]$, $\mu_{\min} > 0$.

EXAMPLE 2.1 (Lowest premiums and highest starting capital). The control with α_2-level starting capital and with lowest premiums[6],

$$\breve{u}(w) \equiv u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), \quad \breve{c}(w) \equiv c_{\min}, \tag{9}$$

where $c_{\min} = \mu_{\min}$, satisfies the (α_1, α_2)-solvency criterion sharply. Indeed,

$$\boldsymbol{\psi}_t(\breve{u}(w), \breve{c}(w), \mu_{\alpha_1}) \equiv \boldsymbol{\psi}_t(u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), c_{\min}, \mu_{\alpha_1}) = \alpha_2$$

by definition of $u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1})$. By Theorem 2.1, the control (9) satisfies the $(\alpha_1+\alpha_2)$-level integral solvency criterion.

This control implies borrowing and "freezes" the insurer's capital. It seems to undercharge the insureds, which contradicts the principle of equity even in its most primitive form: "no premium — no insurance". An additional disadvantage consists in the following. Any sensible control must hold risk reserve large enough to secure solvency, but at the expectation. Otherwise, one would rightfully argue that it is used to cover the unexpected. Taxation of the risk reserves and capitals reflects this requirement, being larger in the latter case. It makes raising capital more expensive than, e.g., holding equalization reserves (see [**4**]).

EXAMPLE 2.2 (Highest premiums and lowest starting capital). The opposite extreme case of hedging against insolvency is yielded by the control with highest premiums[7] and lowest starting capital,

$$\tilde{u}(w) \equiv u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}), \quad \tilde{c}(w) \equiv c_{\max}, \tag{10}$$

where $c_{\max} = \mu_{\alpha_1}$. Again, it satisfies the (α_1, α_2)-solvency criterion sharply: by definition of $u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1})$, one has

$$\boldsymbol{\psi}_t(\tilde{u}(w), \tilde{c}(w), \mu_{\alpha_1}) \equiv \boldsymbol{\psi}_t(u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}), c_{\max}, \mu_{\alpha_1}) = \alpha_2.$$

By Theorem 2.1, the control (10) satisfies the $(\alpha_1 + \alpha_2)$-level integral solvency criterion.

When this control is applied, the insurer's capital is not frozen, but the insureds are severely overcharged, which will not be appreciated by the customers and the regulatory authorities.

Both controls (9) and (10) are rigid (non-adaptive) in the sense that they are not sensitive to the financial results of the previous years and use extensively the premium and the reserve capacities of the insurer.

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[6]Assume that to assign the premium rate less than the least possible value of the claims intensity $\mu_{\min}$ is considered a kind of self-inflicting behavior and is prohibited.

[7]Assume that the highest premium rate $c_{\max}$ can not exceed the upper claims intensity μ_{α_1} because of ethical reasons, or because of restrictions imposed by regulatory authorities.

2.4. "Fair" capital and ultimate equity. The main deficiency of the controls (9) and (10) is they disaccord with the principle of equity. That principle requires "fair" premiums, well-balanced with the claims. In particular, it means that the customers must not be overcharged, but only in the long run: the premiums set in each single insurance year will be inevitably above or below the average. The "fair" long-time average premium rate is $\mathsf{E}M$ since

$$\mathsf{E}V_t(M) = \mathsf{E}M \cdot t, \tag{11}$$

so that the average annual claim amount is equal to the total annual premiums.

Bearing in mind Eq. (8), the initial capital $u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$ is the least necessary to keep the probability of non-ruin within time t equal to $1-\alpha_2$ when the nature selects the worst possible, the largest claim intensity μ_{α_1}, while the insurer is keeping to apply the "long-time-average" premium rate $\mathsf{E}M$. The capital $u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$ may be voted "fair" by those customers who are interested to pay year-by-year the price balanced around the average for their *guaranteed* insurance protection.

We name *equitable* those controls $(u(w), c(w))$ which are holding the risk reserve large enough to secure solvency, but at the expectation i.e., around the "fair" capital value $u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$. Otherwise, one would rightfully argue that this provision is used to cover the unexpected.

Definition 2.5. The adaptive control $(u(w), c(w))$, where w is the past-year-end capital, is called ultimately equitable[8], if

$$\mathsf{E}R_t(u(w), c(w), \mu_{\alpha_1}) = u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$$

uniformly in $w \in \mathsf{R}^+$.

Besides addressing zone-adaptive controls later on in this section, the important device to control deviations of the risk reserve from the expectation at the end of the year is the risk-based dynamic solvency provisions held over many years; it bears analogy to the equalization reserves known in practice.

2.5. Adaptive control satisfying solvency criterion sharply. For $\alpha_i \in (0, 1/2)$, $i = 1, 2$, the adaptive control more sensitive to w than (9) and (10), is

$$\hat{u}(w) = \begin{cases} u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), & w > u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), \\ w, & u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}) \leqslant w \leqslant u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), \\ u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}), & 0 < w < u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}), \end{cases}$$
$$\hat{c}(w) = \begin{cases} c_{\min}, & w > u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), \\ c_{\alpha_2,t}(w, \mu_{\alpha_1}), & u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}) \leqslant w \leqslant u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), \\ c_{\max}, & 0 < w < u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}), \end{cases} \tag{12}$$

where $c_{\min} = \mu_{\min}$, $c_{\max} = \mu_{\alpha_1}$.

Theorem 2.2. *The control $(\hat{u}(w), \hat{c}(w))$ satisfies the (α_1, α_2)-solvency criterion sharply and, consequently, satisfies the $(\alpha_1 + \alpha_2)$-level integral solvency criterion.*

Proof of Theorem 2.2. The proof is straightforward. By definition of $c_{\alpha_2,t}(w, \mu_{\alpha_1})$, for each $u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}) \leqslant w \leqslant u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1})$

$$\boldsymbol{\psi}_t(\hat{u}(w), \hat{c}(w), \mu_{\alpha_1}) = \boldsymbol{\psi}_t(w, c_{\alpha_2,t}(w, \mu_{\alpha_1}), \mu_{\alpha_1}) \equiv \alpha_2,$$

for $w > u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1})$

$$\boldsymbol{\psi}_t(\hat{u}(w), \hat{c}(w), \mu_{\alpha_1}) \equiv \boldsymbol{\psi}_t(u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), c_{\min}, \mu_{\alpha_1}) = \alpha_2,$$

[8] It may be also called balanced around the "fair" capital value $u_{\alpha_2,t}(\;M, \mu_{\alpha_1})$, or targeted at that "fair" capital value.

for $0 < w < u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1})$

$$\boldsymbol{\psi}_t(\hat{u}(w), \hat{c}(w), \mu_{\alpha_1}) \equiv \boldsymbol{\psi}_t(u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}), c_{\max}, \mu_{\alpha_1}) = \alpha_2,$$

and the adaptive control $(\hat{u}(w), \hat{c}(w))$ satisfies the (α_1, α_2)-solvency criterion sharply. The rest of the proof applies Theorem 2.1. □

REMARK 2.3. While $u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}) \leqslant w \leqslant u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1})$, the control (12) does not apply the capital borrowing. When the past-year-end capital w is below $u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1})$, it should be risen. In the opposite case, when w is above $u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1})$, the excess of capital has to be adsorbed, e.g., distributed as dividends. Anticipating the multi-period modelling (see Section 3.3), mention it that wise is to set provisions in the latter case, say, to have them for store in the "years of plenty" in order to cover deficiencies in the former case, say, in the "years of famine"[9].

2.6. Adaptive control with linearized premiums. A technical drawback of the control (12) is the necessity to calculate $c_{\alpha_2,t}(w, \mu_{\alpha_1})$ for each w, i.e., to determine that non-linear function as a whole. Introduce

$$\bar{\tau}_{\alpha_2,t}(w) = -\frac{w - u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})}{t}, \tag{13}$$

where $\mathsf{E}M$ is the ultimately equitable, or "fair" in the sense of Eq. (11), premium rate. Consider the control with linearized adaptive premium rates,

$$\begin{aligned} \bar{u}(w) &= \begin{cases} u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), & w > u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), \\ w, & u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}) \leqslant w \leqslant u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), \\ u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}), & 0 < w < u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}), \end{cases} \\ \bar{c}(w) &= \begin{cases} \mathsf{E}M + \bar{\tau}_{\alpha_2,t}(u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1})), & w > u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), \\ \mathsf{E}M + \bar{\tau}_{\alpha_2,t}(w), & u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}) \leqslant w \leqslant u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), \\ \mathsf{E}M + \bar{\tau}_{\alpha_2,t}(u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1})), & 0 < w < u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}), \end{cases} \end{aligned} \tag{14}$$

where $c_{\min} = \mu_{\min}$, $c_{\max} = \mu_{\alpha_1}$.

On the one hand, calculation of the unique value $u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$ may be easier than calculation of the non-linear function $c_{\alpha_2,t}(w, \mu_{\alpha_1})$. On the other hand, it casts more light on the equity aspects.

The rates $\mathsf{E}M + \bar{\tau}_{\alpha_2,t}(w)$ with the average price component $\mathsf{E}M$ and the adaptive loading $\bar{\tau}_{\alpha_2,t}(w)$, either positive or negative, depend *linearly* on the deviation of the past-year-end risk reserve w from the "fair" capital value $u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$. The case $\bar{\tau}_{\alpha_2,t}(w) > 0$ corresponds to the past-year-end deficit under $u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$, whereas the case $\bar{\tau}_{\alpha_2,t}(w) < 0$ corresponds to the past-year-end surplus over $u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$.

THEOREM 2.3. *The control $(\bar{u}(w), \bar{c}(w))$ is ultimately equitable.*

PROOF OF THEOREM 2.3. When $u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}) \leqslant w \leqslant u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1})$, one has

$$\begin{aligned} R_t(\bar{u}(w), \bar{c}(w), M) &= \bar{u}(w) + \bar{c}(w)\, t - V_t(M) \\ &= z(w) + u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}) + \Big(\mathsf{E}M - \frac{z(w)}{t}\Big)\, t - V_t(M) \\ &= u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}) + \mathsf{E}M \cdot t - V_t(M), \end{aligned}$$

where $z(w) = w - u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$. The similar expression for $R_t(\bar{u}(w), \bar{c}(w), M)$ in two other cases is evident. Bearing in mind Eq. (11), one has

$$\mathsf{E}R_t(\bar{u}(w), \bar{c}(w), \mu_{\alpha_1}) = u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$$

uniformly in $w \in \mathsf{R}^+$. □

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[9] Cf. Bible, Genesis, 41:29 and 41:30.

REMARK 2.4. The insurance years in the diagram (1) are numbered, while the calendar time within each separate insurance year is usually replaced by the so-called *operational* time. It is related to the size of the business portfolio rater than to the real time. It means that the time t in the generic model (2), being operational, may rightfully be assumed large, provided the insurance portfolio is large.

By Theorem 4.4, the lower premium intensity in Eq. (14) is

$$\begin{aligned}\mathsf{E}M + \bar{\tau}_{\alpha_2,t}(u_{\alpha_2,t}(c_{\min},\mu_{\alpha_1})) &= \mathsf{E}M - \frac{u_{\alpha_2,t}(c_{\min},\mu_{\alpha_1}) - u_{\alpha_2,t}(\mathsf{E}M,\mu_{\alpha_1})}{t} \\ &= c_{\min} + \sigma(\mu_{\alpha_1})\,\frac{z_{\alpha_2}((\mu_{\alpha_1}-c_{\min})\sqrt{t}/\sigma(\mu_{\alpha_1})) + z_{\alpha_2}((\mu_{\alpha_1}-\mathsf{E}M)\sqrt{t}/\sigma(\mu_{\alpha_1}))}{\sqrt{t}},\end{aligned}$$

where $z_{\alpha_2}(\cdot)$ is the function introduced in Theorem 4.4, $0 < \kappa_{\alpha_2} \leqslant z_{\alpha_2}((\mu_{\alpha_1} - \mathsf{E}M)\sqrt{t}/\sigma(\mu_{\alpha_1})) \leqslant \kappa_{\alpha_2/2}$, $0 < \kappa_{\alpha_2} \leqslant z_{\alpha_2}((\mu_{\alpha_1} - c_{\min})\sqrt{t}/\sigma(\mu_{\alpha_1})) \leqslant \kappa_{\alpha_2/2}$. For $t \to \infty$, the second summand in the right hand side is tending to zero as $\underline{O}(t^{-1/2})$, and the lower premium intensity in Eq. (14) is close to $c_{\min}$. By the similar arguments, the upper premium intensity in Eq. (14) is close to $c_{\max}$.

The main drawback of the control (14) with linearized premiums is that it satisfies no more the (α_1, α_2)-solvency criterion: the upper bound for the annual probabilities of ruin,

$$\sup_{m \leqslant \mu_{\alpha_1}} \boldsymbol{\psi}_t(\bar{u}(w), \bar{c}(w), m) = \boldsymbol{\psi}_t(\bar{u}(w), \bar{c}(w), \mu_{\alpha_1}),$$

may exceed α_2 for some $w \in \mathsf{R}^+$.

THEOREM 2.4. *One has*

$$c_{\alpha_2,t}(w,\mu_{\alpha_1}) \begin{cases} < \mathsf{E}M + \bar{\tau}_{\alpha_2,t}(w), & w > u_{\alpha_2,t}(\mathsf{E}M,\mu_{\alpha_1}), \\ = \mathsf{E}M + \bar{\tau}_{\alpha_2,t}(w), & w = u_{\alpha_2,t}(\mathsf{E}M,\mu_{\alpha_1}), \\ > \mathsf{E}M + \bar{\tau}_{\alpha_2,t}(w), & 0 < w < u_{\alpha_2,t}(\mathsf{E}M,\mu_{\alpha_1}). \end{cases} \tag{15}$$

PROOF OF THEOREM 2.4. Introduce

$$L(w) = c_{\alpha_2,t}(w,\mu_{\alpha_1}) - (\mathsf{E}M + \bar{\tau}_{\alpha_2,t}(w)), \quad w \geqslant 0, \tag{16}$$

and note that $L(u_{\alpha_2,t}(\mathsf{E}M,\mu_{\alpha_1})) = 0$. It is straightforward from $c_{\alpha_2,t}(u_{\alpha_2,t}(\mathsf{E}M,\mu_{\alpha_1}),\mu_{\alpha_1}) = \mathsf{E}M$ and $\bar{\tau}_{\alpha_2,t}(u_{\alpha_2,t}(\mathsf{E}M,\mu_{\alpha_1})) = 0$ (see Remark 2.2 and Eq. (13)). Theorem 4.4 and Theorem 4.5 yield

$$L(w) = \frac{w}{t} - \frac{\sigma(\mu_{\alpha_1})}{\sqrt{t}} v_{\alpha_2}\Big(\frac{w}{\sigma(\mu_{\alpha_1})\sqrt{t}}\Big) - \frac{\sigma(\mu_{\alpha_1})}{\sqrt{t}} z_{\alpha_2}\Big(\frac{(\mu_{\alpha_1} - \mathsf{E}M)\sqrt{t}}{\sigma(\mu_{\alpha_1})}\Big), \quad w \geqslant 0, \tag{17}$$

where $z_{\alpha_2}(\cdot)$ and $v_{\alpha_2}(\cdot)$ are the functions introduced in Theorems 4.4 and 4.5. Continuous function $L(w)$ is monotone decreasing since $v'_{\alpha_2}(z) > 1$ for $z \geqslant 0$ by Theorem 4.5, and

$$L'(w) = \frac{1}{t}\Big[1 - v'_{\alpha_2}\Big(\frac{w}{\sigma(\mu_{\alpha_1})\sqrt{t}}\Big)\Big] < 0, \quad w \geqslant 0.$$

It completes the proof. □

Theorem 2.4 claims that linearization leads to overcharging the insureds when the past year capital w exceeds the target value $u_{\alpha_2,t}(\mathsf{E}M,\mu_{\alpha_1})$, and to undercharging otherwise. In this sense linearization deteriorates the solvency properties of the control (12) formulated in Theorem 2.2.

2.7. Zone-adaptive control with linearized premiums. Construct a control with linear adaptive loading and free of the drawback of uncontrollable solvency. For the level β such that $0 < \alpha_2 \leqslant \beta < 1/2$, introduce the strip zone with the lower bound $\underline{u}_{\beta,t} = u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}) + z_{\beta,t}$, where $z_{\beta,t} < 0$ is a solution of the equation

$$\boldsymbol{\psi}_t\big(z + u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}), \mathsf{E}M - \frac{z}{t}\,, \mu_{\alpha_1}\big) = \beta \tag{18}$$

with respect to z, and with a certain upper bound $\overline{u}_{\beta,t}$ such that

$$u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}) \leqslant \underline{u}_{\beta,t} \leqslant u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}) \leqslant \overline{u}_{\beta,t} \leqslant u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}).$$

There are different ways to select the upper bound $\overline{u}_{\beta,t}$. For example (recall that $c_{\min} = \mu_{\min}$, $c_{\max} = \mu_{\alpha_1}$), one may take $\overline{u}_{\beta,t} = u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1})$, or[10] $\overline{u}_{\beta,t} = u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$.

Zone-adaptive annual control with linearized premiums is

$$\begin{aligned}
\widehat{u}(w) &= \begin{cases} \overline{u}_{\beta,t}, & w > \overline{u}_{\beta,t}, \\ w, & \underline{u}_{\beta,t} \leqslant w \leqslant \overline{u}_{\beta,t}, \\ \underline{u}_{\beta,t}, & 0 < w < \underline{u}_{\beta,t}, \end{cases} \\
\widehat{c}(w) &= \begin{cases} \overline{\mu}_{\beta,t}, & w > \overline{u}_{\beta,t}, \\ \mathsf{E}M + \overline{\tau}_{\alpha_2,t}(w), & \underline{u}_{\beta,t} \leqslant w \leqslant \overline{u}_{\beta,t}, \\ \underline{\mu}_{\beta,t}, & 0 < w < \underline{u}_{\beta,t}, \end{cases}
\end{aligned} \tag{19}$$

where

$$\overline{\mu}_{\beta,t} = \mathsf{E}M - \frac{\overline{u}_{\beta,t} - u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})}{t},$$
$$\underline{\mu}_{\beta,t} = \mathsf{E}M - \frac{\underline{u}_{\beta,t} - u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})}{t} = \mathsf{E}M - \frac{z_{\beta,t}}{t}.$$

THEOREM 2.5. *For $0 < \alpha_1 < 1/2$, $0 < \alpha_2 \leqslant \beta < 1/2$, the control $(\widehat{u}(w), \widehat{c}(w))$ is ultimately equitable and satisfies the (α_1, β)-solvency criterion sharply.*

PROOF OF THEOREM 2.5. The proof of the first assertion is straightforward. It consists in verification, similarly to the proof of Theorem 2.3, that the equation

$$\mathsf{E}R_t(\widehat{u}(w), \widehat{c}(w), M) = u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$$

holds true uniformly in $w \in \mathsf{R}^+$. The second assertion needs no proof since

$$\sup_{m \leqslant \mu_{\alpha_1}} \boldsymbol{\psi}_t(\widehat{u}(w), \widehat{c}(w), m) = \boldsymbol{\psi}_t(\widehat{u}(w), \widehat{c}(w), \mu_{\alpha_1}) \equiv \beta \tag{20}$$

uniformly in $w \in \mathsf{R}^+$, by Eq. (18). □

THEOREM 2.6. *For $z \in [a, b]$, where $-u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}) < a < 0 < b < \mathsf{E}M \cdot t$, the probability in the left hand side of Eq.* (18) *regarded as a function of z, is monotone decreasing, as z increases.*

PROOF OF THEOREM 2.6. Bearing in mind (3), the proof is straightforward from

$$\begin{aligned}
&\boldsymbol{\psi}_t\big(z + u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}), \mathsf{E}M - \frac{z}{t}, \mu_{\alpha_1}\big) \\
&\qquad = \mathsf{P}\big\{\inf_{0 < s \leqslant t} \big[\big(1 - \frac{s}{t}\big)\, z + (\mathsf{E}M - \mu_{\alpha_1})s - \sigma(\mu_{\alpha_1})\mathsf{W}_s\big] < -u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})\big\},
\end{aligned}$$

[10] That selection is sensible because the premiums will not be larger than M (i.e., $\overline{\mu}_{\beta,t} =$ M in (19)), and no capital exceeding one least necessary to guarantee the non-ruin with probability α_2 is "frozen" as solvency reserve. For $\overline{u}_{\beta,t}$ selected in that way, $|z_{\beta,t}|$ is the width of the strip zone. These reasons may be however unconvincing for a decision maker with other preferences.

since $1 - \dfrac{s}{t} \geqslant 0$ under the infimum sign. □

2.8. Strip width. For $0 < \alpha_1 < 1/2$, $0 < \alpha_2 \leqslant \beta < 1/2$, analyze analytically the existence, the uniqueness and the analytical structure of the solution $z_{\beta,t} < 0$ of Eq. (18) with respect to z.

THEOREM 2.7. *For* $0 < \alpha_1 < 1/2$, $0 < \alpha_2 \leqslant \beta < 1/2$, *put*[11] $z_{\alpha_2,t} = z_{\alpha_2}((\mu_{\alpha_1} - \mathsf{E}M)\sqrt{t}/\sigma(\mu_{\alpha_1}))$, *where* $z_{\alpha_2}(\cdot)$ *is the function introduced in Theorem 4.4. The solution of Eq.* (18) *may be written as*

$$z_{\beta,t} = -\big[(\mu_{\alpha_1} - \mathsf{E}M)t + \sigma(\mu_{\alpha_1})\sqrt{t}\, x_{\beta,t}\big],$$

where $x_{\beta,t} > 0$ *is the unique root of the equation*

$$1 - \Phi(z_{\alpha_2,t}) + \exp\{-2x(z_{\alpha_2,t} - x)\}\Phi(2x - z_{\alpha_2,t}) = \beta. \tag{21}$$

REMARK 2.5. For any $0 < \alpha_2 \leqslant \beta \leqslant 1/2$ and $t \geqslant 0$ the solution of Eq. (21) is bounded from above by a constant, $0 < x_{\beta,t} \leqslant z_{\alpha_2,t} \leqslant \kappa_{\alpha_2/2}$.

PROOF OF THEOREM 2.7. Bearing in mind Theorem 4.4, Theorem 2.6 and Eq. (30), it requires just some direct algebra. □

2.9. Asymptotic analysis and rules of thumb. The results of the previous sections may be summarized as recommendations. It troubleshoots some problems discussed in Introduction and yields certain "rules of thumb" when t is large (see the first paragraph of Remark 2.4).

Even as μ_{α_1} lies not too far from $\mathsf{E}M$, the "fair" or "target" capital value happens to be of order t rather than $\sqrt{t}$, as it was (see Theorem 2.1 in [**9**]) in the case of completely known risk:

$$\begin{aligned} u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}) &= (\mu_{\alpha_1} - \mathsf{E}M)t + \sigma(\mu_{\alpha_1})\sqrt{t}\, z_{\alpha_2,t} \\ &= (\mu_{\alpha_1} - \mathsf{E}M)t + \underline{O}(\sqrt{t}), \quad \text{as } t \to \infty. \end{aligned}$$

Recall that $0 < \kappa_{\alpha_2} \leqslant z_{\alpha_2,t} = z_{\alpha_2}((\mu_{\alpha_1} - \mathsf{E}M)\sqrt{t}/\sigma(\mu_{\alpha_1})) \leqslant \kappa_{\alpha_2/2}$.

The magnitude of the target capital value is of a paramount importance; it is the benchmark for the "long-run mean value", or "appropriate risk-based provisions". That magnitude appears larger for the volatile scenario than for the complete knowledge case dramatically.

The upper bound of the strip zone in (19) is $u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$, while the lower bound is

$$\begin{aligned} \underline{u}_{\beta,t} &= u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}) + z_{\beta,t} \\ &= \sigma(\mu_{\alpha_1})\sqrt{t}[z_{\alpha_2,t} - x_{\beta,t}] = \underline{O}(\sqrt{t}), \quad \text{as } t \to \infty, \end{aligned}$$

and the width of the strip zone is

$$|z_{\beta,t}| = (\mu_{\alpha_1} - \mathsf{E}M)t + \sigma(\mu_{\alpha_1})\sqrt{t}\, x_{\beta,t}.$$

By Remark 2.5, one has $0 < x_{\beta,t} \leqslant z_{\alpha_2,t} = z_{\alpha_2}((\mu_{\alpha_1} - \mathsf{E}M)\sqrt{t}/\sigma(\mu_{\alpha_1})) \leqslant \kappa_{\alpha_2/2}$.

Develop Remark 2.4. For $0 \leqslant \alpha_2 \leqslant 1/2$ and for the capital w_t such that $w_t - \sigma(\mu_{\alpha_1})\sqrt{t} \to +\infty$, as $t \to \infty$, the linearized premium rate $\mathsf{E}M + \bar{\tau}_{\alpha_2,t}(w_t)$ differs from the original premium rate $c_{\alpha_2,t}(w_t, \mu_{\alpha_1})$ by the terms of order $t^{-1/2}$. Deterioration of the original premium rate is therefore rather small in magnitude. By Lemma 4.1, for the function $v_{\alpha_2}(\cdot)$ introduced in Theorem 4.5, one has $v_{\alpha_2}(z) = z - \kappa_{\alpha_2} + \bar{o}(1)$, as $z \to +\infty$. Eq. (17) yields

$$L(w_t) = c_{\alpha_2,t}(w_t, \mu_{\alpha_1}) - \big(\mathsf{E}M + \bar{\tau}_{\alpha_2,t}(w_t)\big) = -\frac{\sigma(\mu_{\alpha_1})}{\sqrt{t}}(z_{\alpha_2,t} - \kappa_{\alpha_2}) + \bar{o}(t^{-1/2}), \quad \text{as } t \to \infty.$$

Bearing in mind that $0 < \kappa_{\alpha_2} \leqslant z_{\alpha_2,t} = z_{\alpha_2}((\mu_{\alpha_1} - \mathsf{E}M)\sqrt{t}/\sigma(\mu_{\alpha_1})) \leqslant \kappa_{\alpha_2/2}$, the right hand side of this equation is $\underline{O}(t^{-1/2})$, as $t \to \infty$. It is also noteworthy that for $0 < \alpha_2 \leqslant \beta \leqslant 1/2$

$$0 < c_{\alpha_2,t}(w_t, \mu_{\alpha_1}) - c_{\beta,t}(w_t, \mu_{\alpha_1}) = \frac{\sigma(\mu_{\alpha_1})}{\sqrt{t}}(\kappa_{\alpha_2} - \kappa_\beta) + \bar{o}(t^{-1/2}), \quad \text{as } t \to \infty.$$

[11] Recall that $\kappa_{\alpha_2/2} = z_{\alpha_2}(0) \geqslant z_{\alpha_2}(v) \geqslant z_{\alpha_2}(+\infty) = \kappa_{\alpha_2} \geqslant 0$ for $v > 0$.

3. Multi-period model of risk under volatile scenario

Recall (see [**7**]–[**9**]) that the rigorous definition of a multi-period controlled risk model over the elementary state space $(\Omega, \mathcal{F})$ with the realizations matching the diagram (1) implies the definition of a controlled random sequence. In the particular case of the

(i) annual mechanisms of insurance (2),
(ii) volatile scenario of nature introduced in Introduction,
(iii) adaptive controls synthesized in Section 2

the "insurer × nature" state space W and the control space U are $\mathsf{R} \times \{0,1\} \times \mathrm{M}$ and $\mathsf{R}^+ \times \mathsf{R}^+$ respectively.

It is noteworthy that since all probability mechanisms of insurance π_k, $k = 1, 2, \ldots$, are assumed[12] to comply with the same generic model (2), we deal with the *homogeneous* multi-period model. It matches well the homogeneous volatile scenario of nature set forth in Definition 1.1.

REMARK 3.1. Discussing long-time average premium rate and long-run mean value of the losses, sensible is to endow the probability mechanisms of insurance π_k with discount factors. It will be done elsewhere, since our concern in this paper is the homogeneous case. Bearing in mind the first paragraph of Remark 2.4, emphasize it that "long run" refers to the number of insurance years rather than to operational lengths of the separate years.

The first component of the state vector $\mathfrak{w}_k = (\mathfrak{w}_k^{\langle 1\rangle}, \mathfrak{w}_k^{\langle 2\rangle}, \mathfrak{w}_k^{\langle 3\rangle}) \in \mathrm{W}$ is the kth year-end capital of the company. The second component indicates whether ruin has occurred, or not, in the kth year. The third component is the outcome of the next-year claims intensity which is the choice of the nature. The two components of the control vector $\mathfrak{u}_{k-1} = (\mathfrak{u}_{k-1}^{\langle 1\rangle}, \mathfrak{u}_{k-1}^{\langle 2\rangle}) \in \mathrm{U}$ are the starting capital and the premium intensity in (2), respectively.

Under certain mild regularity conditions, the controlled random sequence (W_k, M_{k+1}, U_k), $k = 0, 1, \ldots$, assuming values in the product space $(\mathrm{W} \times \mathrm{U}, \mathcal{W} \otimes \mathcal{U})$ is rigorously defined (see, e.g., § 1 of Chapter 1 in [**6**]) by means of $\boldsymbol{\pi} = \{\pi_k,\ k = 1, 2, \ldots\}$ and $\boldsymbol{\gamma} = \{\gamma_k,\ k = 0, 1, \ldots\}$ on a probability space $(\Omega, \mathcal{F}, \mathbf{P}^{\boldsymbol{\pi},\boldsymbol{\gamma}})$.

It is noteworthy that we deal in this paper with Markov (see, e.g., Section 3 of [**9**] for definitions and particulars) annual probability mechanisms of insurance π_k and pure Markov strategies[13] $\boldsymbol{\gamma} = \{\gamma_k,\ k = 0, 1, \ldots\}$. Therefore, the controlled random sequence (W_k, M_{k+1}, U_k), $k = 0, 1, \ldots$, may be reduced to a homogeneous Markov chain on the state space W with the transition probability

$$P(\mathfrak{w}_{k-1}; d\mathfrak{w}_k) = P_{\mathfrak{w}_{k-1}^{\langle 3\rangle}}(\mathfrak{w}_{k-1}^{\langle 1\rangle}; d\mathfrak{w}_k^{\langle 1\rangle} \times d\mathfrak{w}_k^{\langle 2\rangle})\, G(d\mathfrak{w}_k^{\langle 3\rangle}),$$

where

$$\begin{aligned} P_m(\mathfrak{w}_{k-1}^{\langle 1\rangle}; d\mathfrak{w}_k^{\langle 1\rangle} \times \{0\}) &= \mathsf{P}\{R_t(\gamma_{k-1}(\mathfrak{w}_{k-1})) \in d\mathfrak{w}_k^{\langle 1\rangle}, \inf_{0\leqslant s\leqslant t} R_s(\gamma_{k-1}(\mathfrak{w}_{k-1})) > 0 \mid M_k = m\}, \\ P_m(\mathfrak{w}_{k-1}^{\langle 1\rangle}; d\mathfrak{w}_k^{\langle 1\rangle} \times \{1\}) &= \mathsf{P}\{R_t(\gamma_{k-1}(\mathfrak{w}_{k-1})) \in d\mathfrak{w}_k^{\langle 1\rangle}, \inf_{0\leqslant s\leqslant t} R_s(\gamma_{k-1}(\mathfrak{w}_{k-1})) < 0 \mid M_k = m\} \end{aligned} \tag{22}$$

and c.d.f. G is the common distribution of the independent random variables M_k, $k = 1, 2, \ldots$.

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[12]For simplicity's sake. Non-homogeneity, i.e., the case of the annual mechanisms of insurance and the annual controls different within n-years time horizon is a straightforward but a very useful generalization. The unique technical difficulty, though easy to overcome, consists in the non-homogeneous Markov chains which appear in Section 3. An other self-suggesting generalization is the non-homogeneous volatile scenario of nature, i.e., M_k, $k = 1, 2, \ldots$, independent, but not identically distributed.

[13]Or $\boldsymbol{\gamma}_n = \{\gamma_k,\ k = 0, 1, \ldots, n-1\}$, to introduce a more specific notation for the n-years horizon strategy.

Remark 3.2. In the premises of the diffusion model (2), one can easily write the explicit expression for

$$\begin{aligned}P_m(\mathfrak{w}_{k-1}^{\langle 1\rangle}; d\mathfrak{w}_k^{\langle 1\rangle} \times \{0,1\}) &= \mathsf{P}\{R_t(\gamma_{k-1}(\mathfrak{w}_{k-1})) \in d\mathfrak{w}_k^{\langle 1\rangle} \mid M_k = m\} \\ &= \mathsf{P}\{\gamma_{k-1}^{\langle 1\rangle}(\mathfrak{w}_{k-1}) + \gamma_{k-1}^{\langle 2\rangle}(\mathfrak{w}_{k-1})t - (mt + \sigma(m)\mathsf{W}_t) \in d\mathfrak{w}_k^{\langle 1\rangle}\}.\end{aligned}$$

Theorem 4.3 provides the explicit expression for (22) which yields the explicit expression for the transition probability $P(\mathfrak{w}_{k-1}; d\mathfrak{w}_k)$.

We continue to write $\mathbf{P}^{\pi;\gamma}\{\cdot\}$ for the Markov chain with transition probability P and denote by $\mathbf{E}^{\pi;\gamma}$ the mean with respect to that measure. For brevity, we denote by $\mathbf{P}^{\pi;\gamma}_{\boldsymbol{m}}\{\cdot\}$ the conditional distribution $\mathbf{P}^{\pi;\gamma}\{\cdot \mid \boldsymbol{M} = \boldsymbol{m}\}$, where $\boldsymbol{M} = \{M_k,\ k = 1,2,\dots\} \in \mathbf{M} = \mathrm{M}^\infty$ is the sequence of i.i.d. random variables and $\boldsymbol{m}$ is its realization. We denote by $\mathbf{E}^{\pi;\gamma}_{\boldsymbol{m}}$ the respective conditional expectation. Evidently, $\mathbf{P}^{\pi;\gamma}_{\boldsymbol{m}}\{\cdot\}$ corresponds to the case when the trajectory $\boldsymbol{m}$ of the scenario of nature is fixed.

3.1. Solvency.

The following results are fundamental.

Theorem 3.1. *In the homogeneous multi-period Poisson–Exponential model with starting capital $w \in \mathsf{R}^+$, for the homogeneous pure Markov strategy $\boldsymbol{\gamma}$ generated by the annual control* (12),

$$\sup_{w\in\ {}^+} \mathbf{P}^{\pi;\gamma}\left\{\begin{matrix}\text{first ruin in year } k,\\ \text{as starting capital is } w\end{matrix}\right\} \leqslant \alpha_1 + \alpha_2, \quad k = 1,2,\dots, \tag{23}$$

and for the homogeneous pure Markov strategy $\boldsymbol{\gamma}$ generated by the zone-adaptive annual control with linearized premiums (19),

$$\sup_{w\in\ {}^+} \mathbf{P}^{\pi;\gamma}\left\{\begin{matrix}\text{first ruin in year } k,\\ \text{as starting capital is } w\end{matrix}\right\} \leqslant \alpha_1 + \beta, \quad k = 1,2,\dots. \tag{24}$$

Proof of Theorem 3.1. The proof of (23) is immediate from

$$\begin{aligned}\mathbf{P}^{\pi;\gamma}\left\{\begin{matrix}\text{first ruin in year } k,\\ \text{as starting capital is } w\end{matrix}\right\} &= \int_{\times\mathrm{M}} G(dm_1)P_{m_1}(w; d\mathfrak{w}_1^{\langle 1\rangle} \times \{0\}) \dots \\ \dots \int_{\times\mathrm{M}} G(dm_{k-1})\, &P_{\mu_{k-1}}(\mathfrak{w}_{k-2}^{\langle 1\rangle}; d\mathfrak{w}_{k-1}^{\langle 1\rangle} \times \{0\}) \int_{\times\mathrm{M}} G(dm_k)\, P_{m_k}(\mathfrak{w}_{k-1}^{\langle 1\rangle}; \mathsf{R} \times \{1\}),\end{aligned}$$

$$\sup_{\mathfrak{w}_{k-1}^{\langle 1\rangle}\in} \int_{\mathrm{M}} G(dm_k)\, P_{m_k}(\mathfrak{w}_{k-1}^{\langle 1\rangle}; \mathsf{R} \times \{1\}) \leqslant \sup_{\mathfrak{w}_{k-1}^{\langle 1\rangle}\in} \boldsymbol{\psi}_t(\hat{u}(\mathfrak{w}_{k-1}^{\langle 1\rangle}), \hat{c}(\mathfrak{w}_{k-1}^{\langle 1\rangle}), \mu_{\alpha_1}) + \mathsf{P}\{M_k > \mu_{\alpha_1}\}$$

and Theorem 2.2. The proof of (24) is quite analogous and applies Theorem 2.5. □

Corollary 3.1. In the homogeneous multi-period Poisson–Exponential model with starting capital $w \in \mathsf{R}^+$, for the homogeneous pure Markov strategy $\boldsymbol{\gamma}$ generated by the annual control (12),

$$\sup_{w\in\ {}^+} \mathbf{P}^{\pi;\gamma}\left\{\begin{matrix}\text{ruin within } n \text{ years,}\\ \text{as starting capital is } w\end{matrix}\right\} \leqslant \sum_{k=1}^{n} \sup_{w\in\ {}^+} \mathbf{P}^{\pi;\gamma}\left\{\begin{matrix}\text{first ruin in year } k,\\ \text{as starting capital is } w\end{matrix}\right\} \leqslant n(\alpha_1 + \alpha_2)$$

for $n = 1,2,\dots$. For the homogeneous pure Markov strategy $\boldsymbol{\gamma}$ generated by the zone-adaptive annual control with linearized premiums (19), the above inequality holds true with $n(\alpha_1 + \beta)$ instead of $n(\alpha_1 + \alpha_2)$.

3.2. Equity. By Theorem 2.3, in the homogeneous multi-period Poisson–Exponential model with starting capital $w \in \mathsf{R}^+$, the homogeneous pure Markov strategy $\boldsymbol{\gamma}$ generated by the annual control (14) with linearized premiums is equitable in the sense that uniformly in $w \in \mathsf{R}^+$ and for $k = 1, 2, \dots,$

$$\mathsf{E}\left[\mathbf{E}_{\boldsymbol{m}}^{\boldsymbol{\pi},\boldsymbol{\gamma}}\begin{pmatrix}\text{capital at the end of year } k,\\ \text{as starting capital is } w\end{pmatrix}\right] = u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}). \tag{25}$$

That strategy directs the risk reserve at the "target" value $u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$ and makes the risk reserver process balanced around it in a long-time perspective.

The same property holds true for the homogeneous pure Markov strategy $\boldsymbol{\gamma}$ generated by the zone-adaptive annual control with linearized premiums (19).

THEOREM 3.2. *For the homogeneous pure Markov strategy* $\boldsymbol{\gamma}$ *generated by the zone-adaptive annual control with linearized premiums* (19)*, Eq.* (25) *holds true.*

PROOF OF THEOREM 3.2. Note first that for $\boldsymbol{m} = (m_1, m_2, \dots)$

$$\mathbf{E}_{\boldsymbol{m}}^{\boldsymbol{\pi},\boldsymbol{\gamma}}\begin{pmatrix}\text{capital at the end of year } k,\\ \text{as starting capital is } w\end{pmatrix} = \int P_{m_1}(w; d\mathfrak{w}_1^{\langle 1\rangle} \times \{0,1\})$$
$$\dots \int P_{m_{k-1}}(\mathfrak{w}_{k-2}^{\langle 1\rangle}; d\mathfrak{w}_{k-1}^{\langle 1\rangle} \times \{0,1\}) \int \mathfrak{w}_k^{\langle 1\rangle}\, P_{m_k}(\mathfrak{w}_{k-1}^{\langle 1\rangle}; d\mathfrak{w}_k^{\langle 1\rangle} \times \{0,1\}). \tag{26}$$

Bearing in mind Remark 3.2 and Eq. (19), one has

$$\int \mathfrak{w}_k^{\langle 1\rangle}\, P_{m_k}(\mathfrak{w}_{k-1}^{\langle 1\rangle}; d\mathfrak{w}_k^{\langle 1\rangle} \times \{0,1\}) = \int \mathfrak{w}_k^{\langle 1\rangle}\, \mathsf{P}\{\widehat{u}(\mathfrak{w}_{k-1}^{\langle 1\rangle}) + \widehat{c}(\mathfrak{w}_{k-1}^{\langle 1\rangle})t - (m_k t + \sigma(m_k)\mathsf{W}_t) \in d\mathfrak{w}_k^{\langle 1\rangle}\}$$
$$= \begin{cases} (\mathsf{E}M - m_k)t + u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}), & \mathfrak{w}_{k-1}^{\langle 1\rangle} > \overline{u}_{\beta,t},\\ (\mathsf{E}M - m_k)t + u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}), & \underline{u}_{\beta,t} \leqslant \mathfrak{w}_{k-1}^{\langle 1\rangle} \leqslant \overline{u}_{\beta,t},\\ (\mathsf{E}M - m_k)t + u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}), & 0 < \mathfrak{w}_{k-1}^{\langle 1\rangle} < \underline{u}_{\beta,t}.\end{cases}$$

It is noteworthy that the right hand side is independent on $\mathfrak{w}_{k-1}^{\langle 1\rangle}$. Put it in (26). The proof completes by taking expectation over the outcomes $\boldsymbol{m}$ of the scenario of nature. □

REMARK 3.3. The homogeneous pure Markov strategy $\boldsymbol{\gamma}$ generated by the zone-adaptive annual control with linearized premiums (19) is both solvent and equitable.

3.3. Dynamic solvency provisions. Provisions similar to equalization reserves face large deficit at the end of the insurance year. Commonly, these provisions are invested, but in this paper we disregard the investment aspects; one may see that the price which we do not wish to pay for it is more cumbersome transition probabilities.

For zone-adaptive control with $\overline{u}_{\beta,t} = u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$ and with linearized premiums (19) introduce the variable

$$\Delta_t(w) = \begin{cases} 0, & \underline{u}_{\beta,t} \leqslant w \leqslant \overline{u}_{\beta,t},\\ w - \overline{u}_{\beta,t}, & w > \overline{u}_{\beta,t},\\ -(\underline{u}_{\beta,t} - w), & 0 < w < \underline{u}_{\beta,t}.\end{cases}$$

called annual excess (of either sign) of capital. The mean aggregate excess (of either sign) of capital within n years for the strategy $\boldsymbol{\gamma}$, or the mean aggregate dynamic solvency provisions, is

$$\mathsf{E}\left[\mathbf{E}_{\boldsymbol{m}}^{\boldsymbol{\pi},\boldsymbol{\gamma}} \sum_{k=1}^{n} \Delta_t(W_k^{\langle 1\rangle})\right] = \sum_{k=1}^{n} \mathsf{E}\left[\mathbf{E}_{\boldsymbol{m}}^{\boldsymbol{\pi},\boldsymbol{\gamma}} \Delta_t(W_k^{\langle 1\rangle})\right].$$

The following theorem demonstrates that application of the homogeneous pure Markov strategy $\boldsymbol{\gamma}$ generated by the zone-adaptive annual control with linearized premiums (19) increases the mean aggregate dynamic solvency provisions.

THEOREM 3.3. *For the homogeneous pure Markov strategy* $\boldsymbol{\gamma}$ *generated by the zone-adaptive annual control with linearized premiums* (19)*, uniformly in* $w \in \mathsf{R}^+$ *and for* $k = 1, 2, \dots,$

$$\mathsf{E}\Big[\mathbf{E}^{\pi,\gamma}_{\boldsymbol{m}}\Delta_t(W^{\langle 1\rangle}_k)\Big] > 0.$$

PROOF OF THEOREM 3.3. With the starting capital $w \in \mathsf{R}^+$ and $\boldsymbol{m} = (m_1, m_2, \dots)$,

$$\begin{aligned}\mathbf{E}^{\pi,\gamma}_{\boldsymbol{m}}\Delta_t(W^{\langle 1\rangle}_k) = \int P_{m_1}(w; d\mathfrak{w}^{\langle 1\rangle}_1 \times \{0,1\}) \dots \int P_{m_{k-1}}(\mathfrak{w}^{\langle 1\rangle}_{k-2}; d\mathfrak{w}^{\langle 1\rangle}_{k-1} \times \{0,1\}) \\ \times \Big\{ \int_{\{\mathfrak{w}^{\langle 1\rangle}_k > \overline{u}_{\beta,t}\}} \big(\mathfrak{w}^{\langle 1\rangle}_k - \overline{u}_{\beta,t}\big)\, P_{m_k}\big(\mathfrak{w}^{\langle 1\rangle}_{k-1}; d\mathfrak{w}^{\langle 1\rangle}_k \times \{0,1\}\big) \\ - \int_{\{\mathfrak{w}^{\langle 1\rangle}_k < \underline{u}_{\beta,t}\}} \big(\underline{u}_{\beta,t} - \mathfrak{w}^{\langle 1\rangle}_k\big)\, P_{m_k}\big(\mathfrak{w}^{\langle 1\rangle}_{k-1}; d\mathfrak{w}^{\langle 1\rangle}_k \times \{0,1\}\big)\Big\}.\end{aligned} \qquad (27)$$

Bearing in mind Remark 3.2, apply the explicit expression yielded by Theorem 2.1 of [**7**] to the integrand $P_{m_k}(\mathfrak{w}^{\langle 1\rangle}_{k-1}; d\mathfrak{w}^{\langle 1\rangle}_k \times \{0,1\}) = \mathsf{P}\big\{\widehat{u}(\mathfrak{w}^{\langle 1\rangle}_{k-1}) + \widehat{c}(\mathfrak{w}^{\langle 1\rangle}_{k-1})t - (m_k t + \sigma(m_k)\mathsf{W}_t) \in d\mathfrak{w}^{\langle 1\rangle}_k\big\}$ in (27). Direct integration completes the proof. □

4. Auxiliary results

4.1. Mill's ratio and Brownian motion. The most well-known results (see, e.g., [**10**]) for Mill's ratio

$$\mathcal{M}(x) = \frac{1 - \Phi(x)}{\phi(x)} = e^{x^2/2}\int_x^\infty e^{-t^2/2}dt, \quad x \in \mathsf{R}, \qquad (28)$$

are

$$\mathcal{M}(x) > 0, \quad \frac{d}{dx}\mathcal{M}(x) = x\mathcal{M}(x) - 1 < 0, \quad \frac{d^2}{dx^2}\mathcal{M}(x) = \mathcal{M}(x)(1 + x^2) - x > 0, \quad x \in \mathsf{R}, \qquad (29)$$

so that $\mathcal{M}(x)$ is concave and decreasing from ∞ to 0, as x increases from $-\infty$ to ∞. Since

$$\frac{d}{dx}\big(x\mathcal{M}(x)\big) = \frac{d^2}{dx^2}\mathcal{M}(x) > 0, \quad x \in \mathsf{R},$$

the function $x\mathcal{M}(x)$ is increasing from $-\infty$ to 1, as x increases from $-\infty$ to $+\infty$.

For reader's convenience, we collect some formulae for real-valued Brownian motion with linear drift, $\theta t + \sigma\mathsf{W}_t$, $t \geqslant 0$, where $\theta \in \mathsf{R}$, $\sigma > 0$.

THEOREM 4.1. *For* $x \geqslant 0$*, one has* $\mathsf{P}\big\{\sup_{0\leqslant s\leqslant t}\mathsf{W}_s > x\big\} = 2\mathsf{P}\{\mathsf{W}_t > x\}$.

THEOREM 4.2. *For* $x \geqslant 0$ *and* $\theta \in \mathsf{R}$*,* $\sigma > 0$*, one has*

$$\mathsf{P}\Big\{\sup_{0\leqslant s\leqslant t}(\theta s + \sigma\mathsf{W}_s) \leqslant x\Big\} = \Phi\Big(\frac{x - \theta t}{\sigma\sqrt{t}}\Big) - \exp\{2\theta x/\sigma^2\}\Phi\Big(\frac{-x - \theta t}{\sigma\sqrt{t}}\Big).$$

THEOREM 4.3. *For* $x \geqslant 0$ *and* $\theta \in \mathsf{R}$*,* $\sigma > 0$*, one has*

$$\begin{aligned}\mathsf{P}\Big\{\theta t + \sigma\mathsf{W}_t \in dy, \sup_{0\leqslant s\leqslant t}(\theta s + \sigma\mathsf{W}_s) \leqslant x\Big\} = \mathsf{P}\{\theta t + \sigma\mathsf{W}_t \in dy\} \\ - \mathsf{P}\{\theta t + \sigma\mathsf{W}_t \in dy, \sup_{0\leqslant s\leqslant t}(\theta s + \sigma\mathsf{W}_s) \geqslant x\},\end{aligned}$$

where $\mathsf{P}\{\theta t + \sigma\mathsf{W}_t \in dy\} = \dfrac{1}{\sigma\sqrt{2\pi t}}\exp\{-(y - \theta t)^2/2\sigma^2 t\}\,dy$ *and*

$$\mathsf{P}\Big\{\theta t + \sigma\mathsf{W}_t \in dy, \sup_{0\leqslant s\leqslant t}(\theta s + \sigma\mathsf{W}_s) \geqslant x\Big\} = \frac{1}{\sigma\sqrt{2\pi t}}\exp\{(2\theta y t - \theta^2 t^2 - (|y - x| + x)^2)/2\sigma^2 t\}\,dy.$$

These three results are well known. Theorem 4.1 is formula 1.1.4 in Part II, Chapter 1 of [**3**]. For Theorem 4.2 see formula 1.1.4 in Part II, Chapter 2 of [**3**]. For Theorem 4.3 see formulae 1.0.6 and 1.1.8 in Part II, Chapter 2 of [**3**].

4.2. Level values. Bearing in mind the level values introduced in Definition 2.4, address the solution $u_{\alpha,t}(c,m)$ of the equation

$$\boldsymbol{\psi}_t(u,c,m)=\alpha$$

with respect to u and the solution $c_{\alpha,t}(u,m)$ of that equation with respect to c. In the diffusion framework this problem may be solved analytically in a comprehensive way. The generalization of the results of the present paper would require mostly the alternative methods of such analysis, e.g. numerical evaluation or more complicated analytical technique.

THEOREM 4.4. *For $0\leqslant\alpha\leqslant 1/2$, the α-level initial capital corresponding to the claim intensity m is*

$$u_{\alpha,t}(c,m)=\begin{cases}\sigma(m)\sqrt{t}\,\Big[\dfrac{(m-c)\sqrt{t}}{\sigma(m)}+z_\alpha\Big(\dfrac{(m-c)\sqrt{t}}{\sigma(m)}\Big)\Big], & m\geqslant c,\\ \sigma(m)\sqrt{t}\,z_\alpha\Big(\dfrac{(m-c)\sqrt{t}}{\sigma(m)}\Big), & m\leqslant c,\end{cases}$$

where $z_\alpha(v)$ is continuous and monotone increasing, as v increases from $-\infty$ to 0, with

$$0=z_\alpha(-\infty)\leqslant z_\alpha(v)\leqslant z_\alpha(0)=\kappa_{\alpha/2},$$

and monotone decreasing, as v increases from 0 to $+\infty$, with

$$\kappa_{\alpha/2}=z_\alpha(0)\geqslant z_\alpha(v)\geqslant z_\alpha(+\infty)=\kappa_\alpha\geqslant 0.$$

REMARK 4.1. One can supplement Theorem 4.4 by the observation that $u_{\alpha,t}(c,m)$, which depends on m and c only through the difference $m-c$, is a monotone function of this difference. To be more specific, if $m-c$ increases from $-\infty$ to 0, the capital $u_{\alpha,t}(c,m)=u_{\alpha,t}(m-c)$ is monotone increasing from 0 to $\sigma(m)\sqrt{t}\,\kappa_{\alpha/2}$. If $m-c$ increases from 0 to $+\infty$, the capital $u_{\alpha,t}(m-c)$ is monotone increasing from $\sigma(m)\sqrt{t}\,\kappa_{\alpha/2}$ to $+\infty$.

Theorem 4.4 is illustrated by the following numerical calculations.

TABLE 4.1. Values of $u_{\alpha,t}(m-c)$ for $t=100$, $\sigma(m)=1$.

	− = 0	− = 0 01	− = 0 02	− = 0 03	− = 0 04
= 0 3	10 3643	10 6166	10 8823	11 1841	11 5440
= 0 1	16 4485	16 8379	17 2536	17 7159	18 2422
= 0 05	19 5996	20 0331	20 4988	21 0161	21 6000
	− = 0	− = –0 01	− = –0 02	− = –0 03	− = –0 04
= 0 3	10 3643	9 7120	9 0895	8 4983	7 9396
= 0 1	16 4485	15 6601	14 8907	14 1422	13 4161
= 0 05	19 5996	18 7682	17 9517	17 1515	16 3691

THEOREM 4.5. *For $u\geqslant 0$, the α-level premium intensity corresponding to the claim intensity m is*

$$c_{\alpha,t}(u,m)=m-\frac{\sigma(m)}{\sqrt{t}}\,v_\alpha\Big(\frac{u}{\sigma(m)\sqrt{t}}\Big),$$

where $v_\alpha(z)$, $z\geqslant 0$, is continuous, convex and monotone increasing from $-\infty$ to 0, as z increases from 0 to $\kappa_{\alpha/2}$, zero at $z=\kappa_{\alpha/2}$ and monotone increasing from 0 to ∞, as z increases from $\kappa_{\alpha/2}$ to ∞. Furthermore, $v'_\alpha(z)>1$ for $z\geqslant 0$.

Proof of Theorem 4.4. Applying Theorem 4.2, one has

$$\begin{aligned}\psi_t(u,c,m) &= \mathsf{P}\Big\{\inf_{0\leqslant s\leqslant t}[u+(c-m)s-\sigma(m)\mathsf{W}_s]<0\Big\} = \mathsf{P}\Big\{\sup_{0\leqslant s\leqslant t}[(m-c)s+\sigma(m)\mathsf{W}_s]>u\Big\}\\ &= 1-\Phi\Big(\frac{u}{\sigma(m)\sqrt{t}}-\frac{(m-c)\sqrt{t}}{\sigma(m)}\Big)\\ &\qquad + \exp\Big\{2\frac{u}{\sigma(m)\sqrt{t}}\frac{(m-c)\sqrt{t}}{\sigma(m)}\Big\}\Big(1-\Phi\Big(\frac{u}{\sigma(m)\sqrt{t}}+\frac{(m-c)\sqrt{t}}{\sigma(m)}\Big)\Big).\end{aligned} \tag{30}$$

Consider the cases $m \geqslant c$ and $m \leqslant c$ separately. In the former case, bearing in mind (30), rewrite Eq. (8) with respect to u as

$$F_1\Big(\frac{u}{\sigma(m)\sqrt{t}}-\frac{(m-c)\sqrt{t}}{\sigma(m)},\frac{(m-c)\sqrt{t}}{\sigma(m)}\Big)=\alpha, \tag{31}$$

where $F_1(z,v)=1-\Phi(z)+\exp\{2v[v+z]\}(1-\Phi(2v+z))$. The solution $z=z_\alpha(v)$ of the equation

$$F_1(z,v)=\alpha,\quad v\geqslant 0,\ 0\leqslant\alpha\leqslant 1/2,$$

with respect to z exists, is unique, for α fixed is monotone decreasing, as v increases from 0 to $+\infty$, and is bounded, $\kappa_{\alpha/2}=z_\alpha(0)\geqslant z_\alpha(v)\geqslant z_\alpha(+\infty)=\kappa_\alpha\geqslant 0$. For α fixed, $v+z_\alpha(v)$ is monotone increasing, as v increases from 0 to $+\infty$. Moreover, for v fixed, $z_\alpha(v)$ is monotone decreasing, as α increases, and $\infty=z_0(v)\geqslant z_\alpha(v)\geqslant z_1(v)=0$.

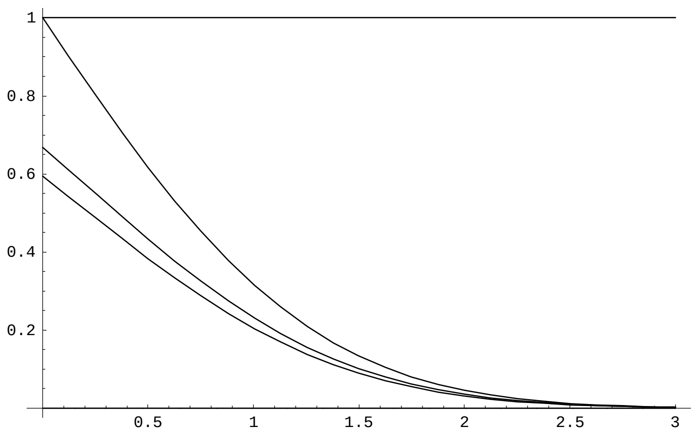

Figure 1. Three graphs: $F_1(z,0)\geqslant F_1(z,1)\geqslant F_1(z,2)$ with $z\geqslant 0$. It is noteworthy that $F_1(0,+\infty)=1/2$.

To prove monotonicity of $z_\alpha(v)$, apply the implicit function derivative theorem and note that[14]

$$\frac{d}{dv}z_\alpha(v)=-\Big(\frac{d}{dv}F_1(z,v)\Big)\Big(\frac{d}{dz}F_1(z,v)\Big)^{-1}\Big|_{z=z_\alpha(v)}=-\frac{2v\mathcal{M}(2v+z)-1}{v\mathcal{M}(2v+z)-1}\Big|_{z=z_\alpha(v)}\leqslant 0$$

[14]Here $\mathcal{M}(x)$, $x\in$, is Mill's ratio, see Eq. (28).

since for $z, v \geqslant 0$

$$\begin{aligned}
\frac{d}{dv}F_1(z,v) &= \exp\{2v[v+z]\}4v(1-\Phi(2v+z)) - 2\exp\{2v[v+z]\}\phi(2v+z) \\
&= 2\exp\{2v[v+z]\}\phi(2v+z)\big[2v\mathcal{M}(2v+z)-1\big] \leqslant 0, \\
\frac{d}{dz}F_1(z,v) &= -\phi(z) + \exp\{2v[v+z]\}2v(1-\Phi(2v+z)) - \exp\{2v[v+z]\}\phi(2v+z) \\
&= 2\exp\{2v[v+z]\}\phi(2v+z)\big[v\mathcal{M}(2v+z)-1\big] \leqslant 0.
\end{aligned}$$

The inequalities $2v\mathcal{M}(2v+z)-1 < 2v/(2v+z)-1 \leqslant 0$ and $v\mathcal{M}(2v+z)-1 < v/(2v+z)-1 \leqslant -1/2$ follow from Eq. (29). Furthermore,

$$\frac{d}{dv}(v+z_\alpha(v)) = 1 + \frac{d}{dv}z_\alpha(v) = 1 - \frac{2v\mathcal{M}(2v+z)-1}{v\mathcal{M}(2v+z)-1}\Big|_{z=z_\alpha(v)} = -\frac{v\mathcal{M}(2v+z)}{v\mathcal{M}(2v+z)-1}\Big|_{z=z_\alpha(v)} \geqslant 0,$$

which yields monotonicity of $v + z_\alpha(v)$. Bearing in mind that $F_1(z,\infty) = 1-\Phi(z)$ and $F_1(z,0) = 2(1-\Phi(z))$, the analysis in the case $m \geqslant c$ is completed.

Address the case $m \leqslant c$. Bearing in mind (30), rewrite Eq. (8) with respect to u as

$$F_2\Big(\frac{u}{\sigma(m)\sqrt{t}}, \frac{(m-c)\sqrt{t}}{\sigma(m)}\Big) = \alpha, \tag{32}$$

where[15] $F_2(z,v) = 1-\Phi(z-v) + \exp\{2zv\}(1-\Phi(z+v))$. The solution $z = z_\alpha(v)$ of the equation

$$F_2(z,v) = \alpha, \quad v \leqslant 0,\ 0 \leqslant \alpha \leqslant 1/2,$$

with respect to z exists, is unique, for α fixed is monotone increasing, as v increases from $-\infty$ to 0, and is bounded, $0 = z_\alpha(-\infty) \leqslant z_\alpha(v) \leqslant z_\alpha(0) = \kappa_{\alpha/2}$. For v fixed, $z_\alpha(v)$ is monotone decreasing, as α increases, and $\infty = z_0(v) \geqslant z_\alpha(v) \geqslant z_1(v) = 0$.

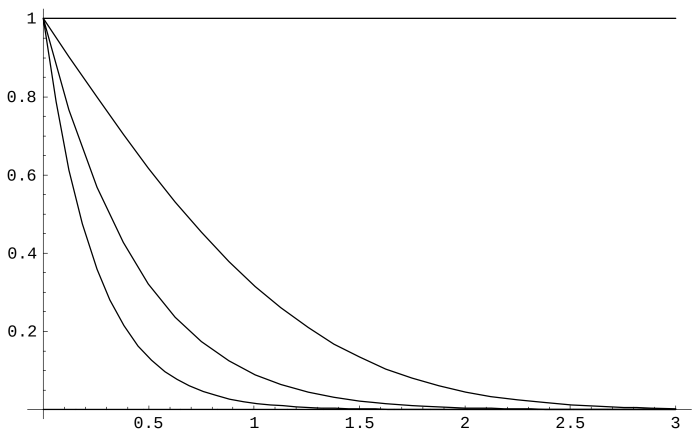

FIGURE 2. Three graphs: $F_2(z,0) \geqslant F_2(z,-1) \geqslant F_2(z,-2)$ with $z \geqslant 0$.

To prove monotonicity of $z_\alpha(v)$, apply the implicit function derivative theorem and note that

$$\frac{d}{dv}z_\alpha(v) = -\Big(\frac{d}{dv}F_2(z,v)\Big)\Big(\frac{d}{dz}F_2(z,v)\Big)^{-1}\Big|_{z=z_\alpha(v)} = -\frac{z\mathcal{M}(z+v)}{v\mathcal{M}(z+v)-1}\Big|_{z=z_\alpha(v)} \geqslant 0$$

[15]One may put $F_2(z,v) = F_1(z-v,v)$ for $v \in$, though our concern is $F_1(z,v)$ for $v \geqslant 0$ and $F_2(z,v)$ for $v \leqslant 0$.

since for $z \geqslant 0$ and $v \leqslant 0$

$$\begin{aligned}\frac{d}{dv}F_2(z,v) &= \phi(z-v) + \exp\{2zv\}2z(1-\Phi(z+v)) - \exp\{2zv\}\phi(z+v)\\ &= \exp\{2zv\}2z(1-\Phi(z+v)) \geqslant 0,\\ \frac{d}{dz}F_2(z,v) &= -\phi(z-v) + \exp\{2zv\}2v(1-\Phi(z+v)) - \exp\{2zv\}\phi(z+v)\\ &= \exp\{2zv\}2v(1-\Phi(z+v)) - 2\exp\{2zv\}\phi(z+v)\\ &= 2\exp\{2zv\}\phi(z+v)[v\mathcal{M}(z+v)-1] < 0.\end{aligned}$$

The inequality $v\mathcal{M}(z+v)-1 < v\mathcal{M}(v)-1 < 0$ is evident from Eq. (29). Bear also in mind that $F_2(z,0) = 2(1-\Phi(z))$. □

Proof of Theorem 4.5. Bearing in mind (30), rewrite Eq. (8) with respect to c as

$$F_2\Big(\frac{u}{\sigma(m)\sqrt{t}}, \frac{(m-c)\sqrt{t}}{\sigma(m)}\Big) = \alpha. \tag{33}$$

The solution $v = v_\alpha(z)$ of the equation

$$F_2(z,v) = \alpha, \quad z \geqslant 0,\ 0 \leqslant \alpha \leqslant 1/2, \tag{34}$$

with respect to v exists, is unique and has the following properties. For α fixed, $v_\alpha(z)$, $z \geqslant 0$, is continuous, convex and monotone increasing from $-\infty$ to 0, as z increases from 0 to $\kappa_{\alpha/2}$, zero at $z = \kappa_{\alpha/2}$ and monotone increasing from 0 to ∞, as z increases from $\kappa_{\alpha/2}$ to ∞. Furthermore, $v'_\alpha(z) > 1$ for $z \geqslant 0$.

To prove monotonicity of $v_\alpha(z)$, apply the implicit function derivative theorem and note that

$$\frac{d}{dz}v_\alpha(z) = -\Big(\frac{d}{dz}F_2(z,v)\Big)\Big(\frac{d}{dv}F_2(z,v)\Big)^{-1}\Big|_{v=z_\alpha(z)} = -\frac{v\mathcal{M}(z+v)-1}{z\mathcal{M}(z+v)}\Big|_{v=v_\alpha(z)} > 1$$

since (see Eq. (29)) for $z \geqslant 0$ and $v \in \mathsf{R}$

$$-\frac{v\mathcal{M}(z+v)-1}{z\mathcal{M}(z+v)} - 1 = \frac{1-(z+v)\mathcal{M}(z+v)}{z\mathcal{M}(z+v)} > 0.$$

The proof is completed. □

Lemma 4.1. *The solution of Eq.* (34) *with respect to v is such that*

$$v_\alpha(z) = z - \kappa_\alpha + \overline{o}(1), \quad \textit{as } z \to +\infty.$$

Proof of Lemma 4.1. Note that

$$\exp\{2zv\}(1-\Phi(z+v)) = \exp\{2zv\}\phi(z+v)\mathcal{M}(z+v) = \phi(z-v)\mathcal{M}(z+v)$$

and rewrite Eq. (34) as

$$1-\Phi(z-v) = \alpha - \phi(z-v)\mathcal{M}(z+v).$$

By Theorem 4.5, $v_\alpha(z) \to +\infty$, as $z \to +\infty$. Since $\mathcal{M}(x) \to 0$, as $x \to +\infty$, and $\phi(x)$ is bounded, the proof is completed. □

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