INSURANCE SOLVENCY AND RISK THEORY

Vsevolod K. Malinovskii

Steklov Mathematical Institute,
119991, Gubkina Str., 8, Moscow, Russia

PART 1:

OVERVIEW

PART 2:

SOME RECENT RESULTS

- $\{T_i\}_{i\geqslant 1}$ are the interclaim times exponentially distributed with parameter $\lambda > 0$
- $\{Y_i\}_{i\geqslant 1}$ are the amounts of claims exponentially distributed with parameter $\mu > 0$

$$R(t) = u + ct - \sum_{i=1}^{N(t)} Y_i, \quad t \geqslant 0,$$

where $u > 0$ is the initial risk reserve, $c > 0$ is the risk premium rate, and $N(t)$ is the largest n for which $\sum_{i=1}^{n} T_i \leqslant t$ (we put $N(t) = 0$ if $T_1 > t$), i.e., $\{N(t)\}_{t\geqslant 0}$ is Poisson process with the rate λ.

$$\tau = (c\,\mathsf{E}T_1 - \mathsf{E}Y_1)/\mathsf{E}Y_1 = c\mu/\lambda - 1$$

is called the relative safety loading, and

$$c = (1+\tau)\lambda/\mu.$$

$$\boldsymbol{\psi}(u;t) = \mathsf{P}\Big\{\inf_{0<s\leqslant t} R(s) < 0\Big\}.$$

Exact formulas

In the Poisson – Exponential risk model

$$\boldsymbol{\psi}(u;t) = \boldsymbol{\psi}(u) - \frac{1}{\pi}\int_0^\pi f(x,u,t)\,dx$$

for any $u > 0$, where

$$\boldsymbol{\psi}(u) = \begin{cases} (\lambda/c\mu)\exp\{-u\mu(1-\lambda/c\mu)\}, & c\mu/\lambda > 1, \\ 1, & c\mu/\lambda \leqslant 1 \end{cases}$$

and

$$f(x,u,t) = (\lambda/c\mu)(1+\lambda/c\mu - 2\sqrt{\lambda/c\mu}\cos x)^{-1} \times \exp\left\{u\mu\Big(\sqrt{\lambda/c\mu}\cos x - 1\Big) - \lambda t(c\mu/\lambda) \times \Big(1+\lambda/c\mu - 2\sqrt{\lambda/c\mu}\cos x\Big)\right\} \times\Big[\cos\Big(u\mu\sqrt{\lambda/c\mu}\sin x\Big) - \cos\Big(u\mu\sqrt{\lambda/c\mu}\sin x + 2x\Big)\Big].$$

$$\boldsymbol{\psi}(u;t) = e^{-u\mu} \sum_{n\geqslant 0} \frac{(u\mu)^n}{n\,!} \left(\frac{\lambda}{c\mu}\right)^{(n+1)/2}$$
$$\times \int_0^{\lambda t} \frac{n+1}{x} e^{-(1+c\mu/\lambda)x} I_{n+1}(2x\sqrt{c\mu/\lambda})\, dx.$$

Let the claim sizes $\{Y_i\}_{i\geqslant 1}$ and the interclaim times $\{T_i\}_{i\geqslant 1}$ be i.i.d. and mutually independent. Let Y_1 be exponential with a parameter $\mu > 0$ and the Laplace transform of T_1 be $\gamma_T(\alpha) = \int_0^\infty e^{-\alpha z} P_T(dz)$. Then

$$\alpha \int_0^\infty e^{-\alpha t} \boldsymbol{\psi}(u;t)\, dt = y(\alpha) \exp\{-u\mu(1 - y(\alpha))\},$$

$\alpha > 0$, where $y(\alpha)$ is a solution of the equation

$$y(\alpha) = \gamma_T(\alpha + c\mu(1 - y(\alpha))), \quad \alpha > 0.$$

Approximations

In Andersen's renewal risk model with $\tau > 0$, set $X_i = Y_i - cT_i$ (i.e., $\mathsf{E}(X_1) < 0$),

$$m_\triangle = \nu^{0,1}_{\bar{X}\bar{T}}(\nu^{1,0}_{\bar{X}\bar{T}})^{-1},$$

$$D^2_\triangle = \mathsf{E}(\nu^{0,1}_{\bar{X}\bar{T}}\bar{X}_1 - \nu^{1,0}_{\bar{X}\bar{T}}\bar{T}_1)^2(\nu^{1,0}_{\bar{X}\bar{T}})^{-3},$$

$$C = \frac{1}{\varkappa\nu^{1,0}_{\bar{X}\bar{T}}}\exp\Big\{-\sum_{n=1}^{\infty}\frac{1}{n}\big(\mathsf{P}\{S_n > 0\} + \mathsf{P}\{\bar{S}_n \leqslant 0\}\big)\Big\},$$

$$t(u) = (t - m_\triangle u)/(D_\triangle u^{1/2}).$$

Let the characteristic function of (T_1, Y_1) be absolutely integrable, $\mathsf{E}(T_1^3) < \infty$ and $D_\triangle > 0$. Then

$$\sup_{t\geqslant 0}\Big|\boldsymbol{\psi}(u;t) - Ce^{-\varkappa u}\Big[\Phi_{(m_\triangle u, D^2_\triangle u)}(t) - Q_1(t(u))\varphi_{(m_\triangle u, D^2_\triangle u)}(t)\Big]\Big| = \overline{o}(e^{-\varkappa u}u^{-1}),$$

as $u \to \infty$.

In the Poisson – Exponential risk model

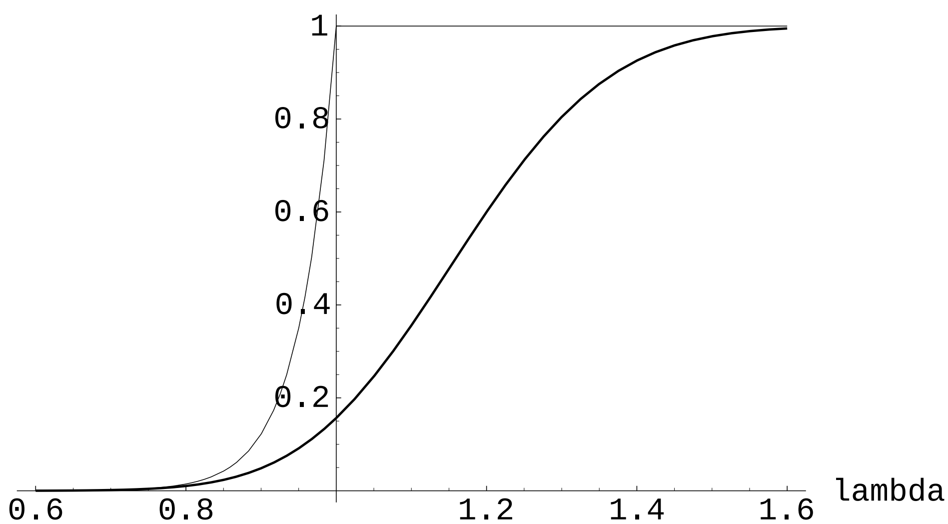
$$\varkappa = \mu\tau/(1+\tau),$$

$$m_\triangle = \mu/(\lambda\tau(1+\tau)), \quad D_\triangle^2 = 2\mu/(\lambda^2\tau^3),$$

$$C = 1/(1+\tau).$$

Malinovskii, V.K. (1994) Corrected normal approximation for the probability of ruin within finite time, *Scandinavian Actuarial Journal*, 161–174.

Malinovskii, V.K. (1998) Non-poissonian claims arrivals and calculation of the probability of ruin, *Insurance: Mathematics and Economics*, vol. 22, 123–138.



Exact values of $\boldsymbol{\psi}(u)$ (thin) and $\boldsymbol{\psi}(u;t)$ (bold) with $u = 20$, $t = 100$, $\mu = c = 1$, $\lambda > 0$.

Since $\tau = (1 - \lambda)/\lambda$, the inequality $\lambda > 1$ is equivalent to $\tau < 0$, while $\lambda < 1$ yields $\tau > 0$.

Bohman, H. (1987) Interview with Harald Bohman[1] (Interviewers: B. Ajne and B. Palmgren), *Scandinavian Actuarial Journal*, 1–3.

"I was for a long time deeply involved in this theory, working on the probability of ruin, but I am hesitant over it now. There has been so much theoretical work and so little practical result. At the beginning of my career, the Swedish pioneers were still active: Filip Lundberg, Carl-Otto Segerdahl, Bertil Almer, Fredrik Esscher, and Carl Philipson. It was then said: "We have the theory, we lack numerical results". Later on the Swedish "Convolution

[1]Farewell interview as retiring Chief Editor of the Scandinavian Actuarial Journal.

Committee" was organised, and it prepared numerical results from empirical data on claims distributions. Theoretical improvements were made by Olof Thorin and illustrated numerically by Nils Wikstad.

Now that we have the numerical results, the discussion about solvency and consolidation has not abated. Thus, I conclude, that from a practical point of view, the theory of collective risk, as initiated by Filip Lundberg, has missed the point, because the underlying model is unrealistic, too simplified. For one thing, a stationary business should give stationary reserves, as predicted by the control theory. It must be admitted, though, that the model has had an enormous success from a theoretical point of view, judging from the number of scientific papers devoted to it." (p. 2)

Cramér, H. (1969) Historical review of Filip Lundberg's works on risk theory, *Skandinavisk Aktuarietidskrift*, *Suppl.*, 6–12.

"In view of certain misconceptions that have appeared it is, however, necessary to point out that Lundberg repeatedly emphasizes the practical importance of some arrangement which automatically prevents the risk reserve from growing unduly. This point is, in fact, extensively discussed in the papers of 1909, 1919 and 1926 – 28. One possible arrangement proposed to this end is to work with a security factor $\tau = \tau(x)$ which is a decreasing function of the risk reserve $R(t) = x$. Another possibility is to dispose, at predetermined epochs, of part of the risk reserve for bonus distribution. By either method, the growth of the risk reserve may be efficiently controlled. What Lundberg does in this connection is really to work with a rather refined case of what has much later come to be known as a random walk with two barriers.

From certain quarters, the Lundberg's theory has been declared to be unrealistic because, it is asserted, no limit is imposed on the growth of the risk reserve. In view of what has been said above, it would seem that these critics have not read the author they are criticizing. For a non-Scandinavian author there is, of course, the excuse that most of Lundberg's works are written in Swedish."

Borch, K. (1967) The theory of risk, *Journal of the Royal Statist. Soc., Ser. B*, vol. 29, no. 3, 432–452; Discussion, *ibid.*, 452–467.

"The most unrealistic assumptions in the models we have discussed seem to be:

(i) The stationarity assumptions, which imply that the nature of the company's business will never change. These assumptions become less drastic than they may seem at first sight, if we introduce operational time.

(ii) The assumption that the probability laws governing the process are completely known.

(iii) The implicit assumption that a decision once it has been made cannot be changed."(p. 450)

Borch, K. (1967) The theory of risk, *Journal of the Royal Statist. Soc., Ser. B*, vol. 29, no. 3, 432–452; Discussion, *ibid.*, 452–467.

"the possible generalizations of the risk theory . . . should lead to models which contain all the essential elements of the real problems in insurance companies. The models are, however, so general that they can be given a number of other interpretations, and applied to a wide range of practical problems in different fields." (p. 451)

Pentikäinen, T. (1975) A model of stochastic-dynamic prognosis. An application of risk theory to business planning, *Scandinavian Actuarial Journal*, 29–53.

"Our purpose is to attack just this problem and to endeavor to build up a picture of the management process of the insurance business in its entirety (as far as possible) and to place the risk theoretical aspects in it as a part among numerous other parts, most of which are not of an actuarial character. In this way some of the classical applications of risk theory are amalgamated with the ideas of modern business planning, especially with the technics of long-range prognoses on the basis of different, often alternative preassumptions or, as it is often called, different business strategies." (p. 29)

Emerging costs format transition equation (equation (1.1.1) in: Daykin, C.D., Pentikäinen, T., Pesonen, M. (1996) *Practical Risk Theory for Actuaries.* Chapman and Hall, London, etc.)

$$R_k = R_{k-1} + I_k + C_k + V_k^{\mathrm{re}} + A_k^{\mathrm{new}} + B_k^{\mathrm{new}} - V_k - E_k - I_k^{\mathrm{re}} - D_k.$$

Effective period is an account year, so that k is an integer variable,

R_k — the amount of assets at the end of k-th period,

R_{k-1} — the initial amount,

I_k — the premium income,

C_k — the return received in respect of the investments during the k-th period,

V_k^{re} — the recovery from reinsurers during that period,

A_k^{new} — the new equity capital issued and subscribed for during that period,

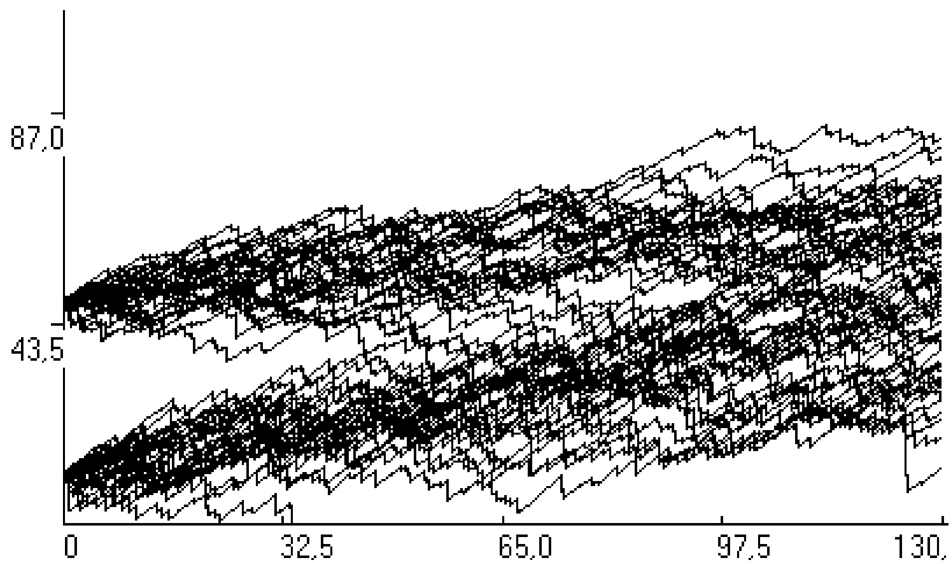
B_k^{new} — the new debt capital issued and subscribed for during the k-th period and any other borrowing,

V_k — the amount of claim payments made during the k-th period including payments made on account,

E_k — the amount of commission paid and administration and operation expenses in the k-th period,

I_k^{re} — ceded reinsurance premium in the k-th period,

D_k — the dividends paid to shareholders and bonuses paid to policyholders in the k-th period.



Two simulated bundles for the Poisson – Exponential risk model with $\lambda = \mu = 1$; lower: $u = 10$, upper: $u = 50$, for both $c = 1 + \tau^{\{5\}}(u)$, $\tau^{\{5\}}(u) = 2.15534\, u^{-11/12}$, i.e., $\tau^{\{5\}}(10) = 0.26113$ and $\tau^{\{5\}}(50) = 0.05972$.

Directive 2002/13/EC of the European Parliament and of the Council of 5 March 2002, Brussels, 5 March 2002.

(Simplified algorithm as specified in the Article 16 [a]). The amount of assets R_k at the end of k-th year is subject to the transition equation

$$R_k = U_{k-1} + I_k - V_k, \quad k = 1, 2, \dots,$$

U_{k-1} is the capital at the beginning of k-th year,

$$U_{k-1} = \begin{cases} \min\{R_{k-1}, \alpha_{k-1} z_{k-1}\}, & R_{k-1} \geqslant z_{k-1}, \\ z_{k-1}, & R_{k-1} < z_{k-1}, \end{cases}$$

z_{k-1} is the solvency margin calculated on the base of past three financial years,

$$z_{k-1} = \beta_{k-1} \max\Big\{0.18\, I_{k-1}, \\ 0.26\, \tfrac{1}{3}[V_{k-1} + V_{k-2} + V_{k-3}]\Big\},$$

and $\alpha_{k-1} \geqslant 1$, $\beta_{k-1} \geqslant 1/2$.

Bible, Genesis 41

41:29 Behold, there come seven years of great plenty throughout all the land of Egypt:
41:30 And there shall arise after them seven years of famine; and all the plenty shall be forgotten in the land of Egypt; and the famine shall consume the land;
41:31 And the plenty shall not be known in the land by reason of that famine following; for it shall be very grievous.

41:33 Now therefore let Pharaoh look out a man discreet and wise, and set him over the land of Egypt.
41:34 Let Pharaoh do this, and let him appoint officers over the land, and take up the fifth part of the land of Egypt in the seven plenteous years.
41:35 And let them gather all the food of those good years that come, and lay up corn under the hand of Pharaoh, and let them keep food in the cities.
41:36 And that food shall be for store to the land against the seven years of famine, which shall be in the land of Egypt; that the land perish not through the famine.

41:47 And in the seven plenteous years the earth brought forth by handfuls.
41:48 And he gathered up all the food of the seven years, which were in the land of Egypt, and laid up the food in the cities: the food of the field, which was round about every city, laid he up in the same.
41:49 And Joseph gathered corn as the sand of the sea, very much, until he left numbering; for it was without number.

41:54 And the seven years of dearth began to come, according as Joseph had said: and the dearth was in all lands; but in all the land of Egypt there was bread.
41:55 And when all the land of Egypt was famished, the people cried to Pharaoh for bread: and Pharaoh said unto all the Egyptians, Go unto Joseph; what he saith to you, do.
41:56 And the famine was over all the face of the earth: And Joseph opened all the storehouses, and sold unto the Egyptians; and the famine waxed sore in the land of Egypt.

Borch, K. (1967) The theory of risk, *Journal of the Royal Statist. Soc., Ser. B*, vol. 29, no. 3, 432–452; Discussion, *ibid.*, 452–467.

"We have now reached the point where the actuarial theory of risk again joins the mainstream of theoretical statistics and applied mathematics. Our general formulation of the actuary's problem leads directly to the general theory of *optimal control processes* or *adaptive control processes* ...

The theory of control processes seems to be "tailor-made" for the problems which actuaries have struggled to formulate for more than a century." (p. 451)

$$w_0 \underbrace{\xrightarrow{\gamma_0} u_0 \xrightarrow{\pi_1} w_1}_{\text{1-st year}} \cdots \xrightarrow{\pi_{k-1}} w_{k-1} \underbrace{\xrightarrow{\gamma_{k-1}} u_{k-1} \xrightarrow{\pi_k} w_k}_{k\text{-th year}} \cdots$$

- the annual transition functions

$$\pi_k(w_0, \ldots, w_{k-1}, u_0, \ldots, u_{k-1}; dw_k), \quad k = 1, 2, \ldots,$$

- the annual control rules

$$\gamma_{k-1}(w_0, \ldots, w_{k-1}, u_0, \ldots, u_{k-2}; du_{k-1}), \quad k = 1, 2, \ldots$$

Some examples

$$\pi_k(u_{k-1}; dw_k \vdash t_k) = \mathsf{P}\Big\{R_k(t_k) \in dw_k \mid R_k(0) = u_{k-1}\Big\};$$

$$\pi_k(u_{k-1}; dw_k \vdash t_k, \theta_k) = \mathsf{P}_{\theta_k}\Big\{R_k(t_k; \check{\theta}_k) \in dw_k^{\langle 1 \rangle}, \hat{\theta}_k(t_k) \in dw_k^{\langle 2 \rangle} \mid R_k(0) = u_{k-1}\Big\},$$

where $dw_k = dw_k^{\langle 1 \rangle} \times dw_k^{\langle 2 \rangle}$ (feed-back and feed-forward).

Pentikäinen, T. (1975) A model of stochastic-dynamic prognosis. An application of risk theory to business planning, *Scandinavian Actuarial Journal*, 29–53.

"One great advantage of the analytic method, even if it is based on very special assumptions, is that the interdependence of the variables involved can be illustrated. Even if the values obtained are far from the values obtained by the original assumptions, probably at least the shape of the interdependence can be preliminary studied in this way which makes it easier to understand the structure of the complicated model." (p. 45)

Malinovskii, V.K. (2002) On risk reserve conditioned by ruin. Contribution to: 27-th International Congress of Actuaries, Cancún, México, 17–22 March, 2002.

In Andersen's renewal risk model with $\tau > 0$, for

$$m(t) = \inf_{0<s\leqslant t} R(s)$$

and under certain regularity conditions,

$$\sup_{t\geqslant 0,\ x\in\mathsf{R}} \Big| e^{\varkappa u}\mathsf{P}\{R(t)\leqslant x, m(t)<0 \mid R(0)=u\} - C\int_0^t \varphi_{(m_\triangle u, D^2_\triangle u)}(z)\varPhi_{(m[t-z],D^2[t-z])}(x)\,dz\Big| \to 0,$$

as $u\to\infty$.

Malinovskii, V.K. (2000) Probabilities of ruin when the safety loading tends to zero, *Advances in Applied Probability*, vol. 32, 885–923.

In Andersen's renewal risk model the premium rate c_u is called *asymptotically reduced of order* τ_u, if

$$c_u = (1+\tau_u)\mathsf{E}(Y_1)/\mathsf{E}(T_1),$$

with $\tau_u > 0$ and $\tau_u \to 0$ monotonously, as $u \to \infty$.

Suppose that c_u is asymptotically reduced of order $\tau_u \geqslant u^{-5/12}$ and certain regularity conditions hold true. Then, as $u \to \infty$,

$$\sup_{t \geqslant 0} \left| \boldsymbol{\psi}(u;t) - C_u e^{-\varkappa_u u} \Phi_{(m_u u, D_u^2 u)}(t) \right| = \underline{O}\Big((\tau_u u)^{-1/2} e^{-\varkappa_u u}\Big).$$

(in P/E case $\varkappa_u = \mu\tau_u/(1+\tau_u)$, $C_u = 1/(1+\tau_u)$, $m_u = \mu/(\lambda\tau_u(1+\tau_u))$, $D_u^2 = 2\mu/(\lambda^2\tau_u^3)$.)

Set

$$R_t(u,\tau) = u + ct - \sum_{i=1}^{N(t)} Y_i, \quad t \geqslant 0.$$

The **"target"** value $u_{\alpha,t}$ of the risk reserve corresponding to the probability of ruin $\alpha \in (0,1)$ is the value $u = u_{\alpha,t}$ which satisfies the equation

$$\psi_t(u_{\alpha,t}, 0) = \alpha.$$

In the Poisson – Exponential risk model, the target value $u_{\alpha,t}$ corresponding to the probability of ruin $\alpha \in (0,1)$ is the solution of the equation

$$\int_0^\pi f_t(x; u, 0)dx = \pi(1-\alpha),$$

where

$$f_t(x;u,0) = (2(1-\cos x))^{-1} \exp\left\{(\mu\cos x - 1)(u + 2\lambda t)\right\}\left[\cos(u\mu\sin x) - \cos(u\mu\sin x + 2x)\right].$$

1. Control without borrowing

Set $u_z = u_{\alpha,t} + z$ and $\tau_{z,t} = -\dfrac{\mu}{\lambda t} z$. Then

$$\mathsf{E} R_t(u_z, \tau_{z,t}) = u_{\alpha,t} \quad \text{for any} \quad z.$$

2. Control with borrowing

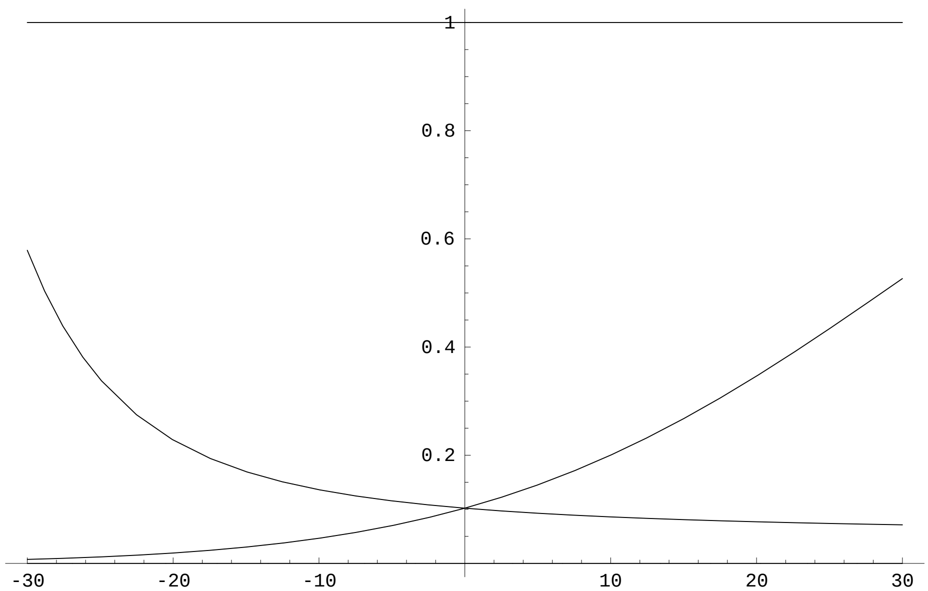
Set

$$u_z = \begin{cases} u_{\alpha,t}, & z \leqslant 0, \\ u_{\alpha,t} + z, & z > 0 \end{cases} \quad \text{and} \quad \tau_{z,t} = -\frac{\mu}{\lambda t} z.$$

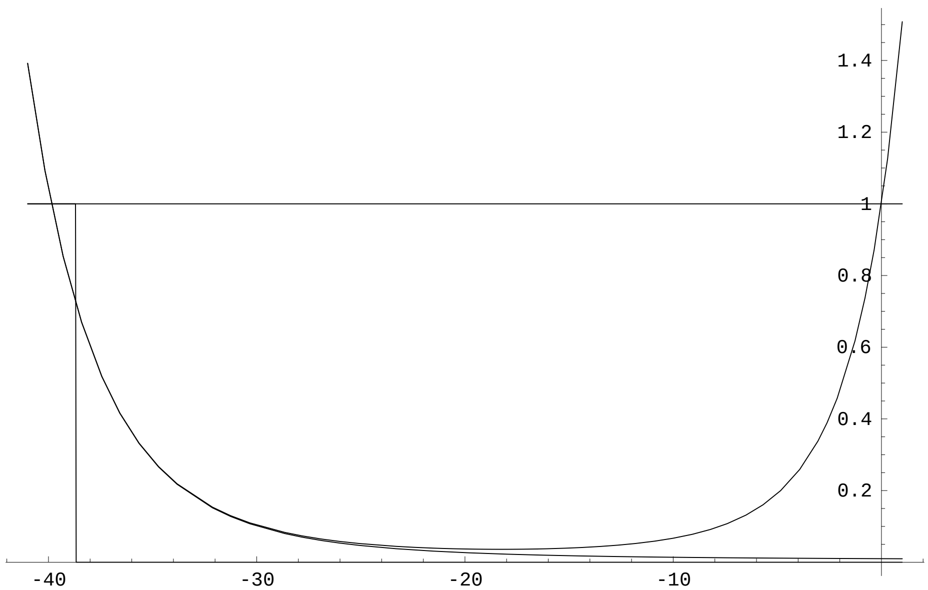
Then

$$\mathsf{E} R_t(u_z, \tau_{z,t}) = \begin{cases} u_{\alpha,t} - z, & z \leqslant 0, \\ u_{\alpha,t}, & z > 0, \end{cases}$$

and in case $z \leqslant 0$ (deficit), when $|z|$ must be borrowed at the beginning of the insurance year, $|z|$ equals to the average surplus at the end of the year; it repays the borrowing.



Two finite time ($t = 200$, $\mu = \lambda = 1$) ruin probabilities $\psi_t(u_z, \tau_{z,t})$ as functions of the capital deviation z. Increasing graph: $u_z \equiv 33$, $\tau_{z,t} = -z/t$. Decreasing graph: $u_z = 33 + z$, $\tau_{z,t} = -z/t$.



Ruin probability $\psi_t(u_z, \tau_{z,t})$, $u_z = u + z$, $\tau_{z,t} = -z/t$, as function of the capital deviation z, with $t = 100$, $\mu = \lambda = 1$, $u = 38.6811$, so that $\alpha = 0.01$, and the upper bound.