ON RISK RESERVE CONDITIONED BY RUIN

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Summary

The distribution of the risk reserve at time t conditional on ruin within time t is considered in Andersen's collective risk model. Approximations for large initial capital and certain numerical results are presented.

The problem is motivated by the wish to get more insight on the consequences of ruin during the time interval $(0, t]$. In particular, what would be the capital of a company at the end of the accounting period if, once ruin has occurred, the insurer's usual activities continued during this period, including the acceptance of new business (a sort of going-concern philosophy).

LA RÉSERVE DE RISQUE CONDITIONNELLE À LA RUINE

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Résumé

La distribution de la réserve de risque au temps t conditionnelle à la ruine dans l'intervalle $(0, t]$ est considérée dans le modèle collectif de risque d'Andersen. Des approximations lorsque la réserve de risque initiale est grande sont introduites et quelques illustrations numériques sont présentées.

Ce problème est motivé par le désir d'obtenir une plus grande compréhension des conséquences de la ruine durant l'intervale de temps $(0, t]$. Notamment, quel sera le capital d'une compagnie vers la fin de la période de comptabilité si, une fois que la ruine a eu lieu, l'assureur continue ses activités durant toute cette période et accepte de nouveaux contrats (une sorte de philosophie de devenir inquiet).

1. Introduction

The purpose of this article is to present mathematical results concerning correlation between ruin within time t and the insurer's surplus at time t in the collective risk model with light tailed claims and positive safety loading. Motivated by the question of what the impact of the ruin on the surplus of the insurer will be, we want to explore the distribution and the moments of the surplus at the end of the interval $(0, t]$ conditioned by ruin, provided the business conditions after the ruin remain unchanged except, possibly, alteration of the premium rate due to first ruin shock.

This question is of theoretical and of a certain practical interest. The ruin is not equivalent to insolvency of the insurer. It is an element of an early warning system for guidance of an insurance project. Some projects are allowable to relatively high probability of ruin. Even the negative surplus at certain times may not be a sufficient evidence against the inward profitable line of business or the insurance project if the damage will be covered by the end of the planning period. This analysis of after-ruin is of interest e.g., for companies with extra capital, seeking to gain a market share by cutting prices. The underwriting may result in underreserving in this case, with conscious losses, and even occasional exhausting of the initially allocated reserves.

Our interest is motivated also by a desire to generalize the collective risk model over consecutive business years or uneven time periods between premium rates adjustments. The former depends on supervision which requires the submission of financial reports on operations on a yearly basis. The later reflects changes in the world of applications which give rise to the necessity of examining the amount of surplus on a periodic basis. Bearing in mind the time needed for development and implementation of amended tariffs, these uneven time periods frequently are taken on a semiannual, quarterly, or monthly basis. The area of application of this model seems to have little intersection with the collective risk model where immediate changes of the premium rate are allowed depending on the value of the risk reserve (see e.g., Asmussen, Nielsen (1995)).

Mathematically, the problem consists in generalizing of the classical Cramér–Lundberg approximation for the finite time ruin probability $\psi(t,u)$,

$$\lim_{u\to\infty}\sup_{t\geqslant 0}\left|e^{\varkappa u}\psi(t,u)-C\Phi_{(m_1u,D_1^2u)}(t)\right|=0,\quad \text{as}\quad u\to\infty, \tag{1}$$

(see e.g., von Bahr (1974)) which reveals the expression which approximates the probability that an intrinsically profitable insurer falls into ruin. The joint distribution of the time of ruin, the surplus immediately before ruin, and the deficit at ruin were also studied (see e.g., Gerber, Shiu (1998)). We approach the question of how much the event of ruin endamages the insurer's capital at the end of the accounting period in case he is allowed to operate after ruin.

2. Notation and assumptions

Andersen's risk model comes from the i.i.d. random vectors $\{(Y_i,T_i)\}_{i\geqslant 1}$, where T_i are the interclaim times and Y_i are the amounts of claims, with the probability distribution function (p.d.f.) $B_{Y,T}(y,t)=\mathbf{P}\{Y_1\leqslant y, T_1\leqslant t\}$ and the characteristic function (ch.f.) $\beta_{Y,T}(t_1,t_2)=\mathbf{E}\exp(it_1Y_1+it_2T_1)$. These random vectors generate the risk reserve process

$$R_u(t)=u+c[t\wedge\omega]+b[t-t\wedge\omega]-\sum_{i=1}^{N(t)}Y_i,\quad t\geqslant 0, \tag{2}$$

where $u>0$ is the initial risk reserve, ω denotes the time of the first ruin, $t\wedge\omega=\min\{t,\omega\}$, $c>0$ is the risk premium rate before the first ruin, b is the risk premium rate after the first ruin (of course, it may be equal to c which means no premium rate change after the first ruin), and $N(t)$ is the largest n for which $\sum_{i=1}^{n}T_i\leqslant t$

(we put $N(t) = 0$ if $T_1 > t$). Consider

$$c = (1+\tau)\mathbf{E}Y_1/\mathbf{E}T_1, \tag{3}$$

with τ called relative safety loading and assume $\tau > 0$. Evidently, the equality (3) is equivalent to $\tau = c\mathbf{E}T_1/\mathbf{E}Y_1 - 1$.

Ruin occurs at time s as $R_u(s) < 0$ and the probability that ruin occurs within the time interval $(0,t]$ is $\psi(t,u) = \mathbf{P}[\inf_{0<s\leqslant t} R_u(s) < 0]$.

For $i = 1,2,\dots$ introduce i.i.d. random variables $X_i = Y_i - cT_i$ and put $S_n = \sum_{i=1}^n X_i$, $V_n = \sum_{i=1}^n Y_i$. For the p.d.f. $B(x,y) = \mathbf{P}\{X_1 \leqslant x, T_1 \leqslant y\}$ and for a positive solution $\varkappa$ of the Lundberg equation,

$$\mathbf{E}\exp(\varkappa X_1) = 1, \tag{4}$$

introduce an associate p.d.f. by $\overline{B}(dx,dy) = e^{\varkappa x}B(dx,dy)$. For notational convenience, introduce the associated sequence $\{(\overline{X}_i, \overline{T}_i)\}_{i\geqslant 1}$ of i.i.d. random vectors having the p.d.f. $\overline{B}(x,y)$, and $\overline{S}_n = \sum_{i=1}^n \overline{X}_i$, $\overline{U}_n = \sum_{i=1}^n \overline{T}_i$. Put

$$\nu^{i,j} = \mathbf{E}Y_1^i T_1^j, \quad \overline{\nu}^{i,j} = \mathbf{E}\overline{X}_1^i \overline{T}_1^j, \quad i,j = 0,1\dots\ . \tag{5}$$

For real b introduce

$$\begin{aligned}
m_1 &= \overline{\nu}^{0,1}/\overline{\nu}^{1,0}, \quad m_2(b) = b - \nu^{1,0}/\nu^{0,1}, \\
D_1^2 &= ((\overline{\nu}^{0,1})^2\overline{\nu}^{2,0} - 2\overline{\nu}^{1,0}\overline{\nu}^{0,1}\overline{\nu}^{1,1} + (\overline{\nu}^{1,0})^2\overline{\nu}^{0,2})/(\overline{\nu}^{1,0})^3, \\
D_2^2 &= ((\nu^{0,1})^2\nu^{2,0} - 2\nu^{1,0}\nu^{0,1}\nu^{1,1} + (\nu^{1,0})^2\nu^{0,2})/(\nu^{0,1})^3, \\
C &= \frac{1}{\varkappa\overline{\nu}^{1,0}}\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\left[\mathbf{P}(S_n > 0) + \mathbf{P}(\overline{S}_n \leqslant 0)\right]\right),
\end{aligned} \tag{6}$$

and for the Normal distribution and density functions $\Phi_{(\mu,\sigma^2)}(z)$ and $\varphi_{(\mu,\sigma^2)}(z)$ introduce

$$g(z) = z + \varphi_{(0,1)}(z)\Phi^{-1}_{(0,1)}(z). \tag{7}$$

Using the Mill's relation, we have $g(z) = z(1+\overline{o}(1))$, as $z \to \infty$, and using the approximation $\Phi_{(0,1)}(z) = \frac{1}{2} + \varphi_{(0,1)}(z)(z + \frac{1}{3}z^3 + \dots)$, we have $g(z)$ approximated by $(1 - 4\varphi^2_{(0,1)}(z))z + 2\varphi_{(0,1)}(z)$ for z in a neighborhood of zero.

3. Approximations

Introduce

$$\psi(w;t,u) = \mathbf{P}\big[R_u(t) \leqslant w, \inf_{0<s\leqslant t} R_u(s) < 0\big].$$

Evidently, $\psi(+\infty;t,u) = \psi(t,u)$.

Theorem 1. *Suppose that in the collective risk model with $\tau > 0$ the characteristic function $\beta_{Y,T}(t_1,t_2)$ is absolutely integrable and $0 < D_1, D_2 < \infty$. Suppose that up to the first ruin the premium rate c is as in (3) and after ruin it becomes b. Then, as $u \to \infty$,*

$$\lim_{u\to\infty} \sup_{t\geqslant 0, w\in\mathbf{R}} \left| e^{\varkappa u}\psi(w;t,u) - C\int_0^t \varphi_{(m_1 u, D_1^2 u)}(z)\Phi_{(m_2(b)[t-z], D_2^2[t-z])}(w)\,dz \right| = 0.$$

The approximation (1) is a particular case of one in Theorem 1.

Theorem 2. *Under the conditions of Theorem 1*

$$\mathbf{E}\big[R_u(t) \mid \inf_{0<s\leqslant t} R_u(s) < 0\big] = m_2(b) D_1\sqrt{u}\, g\left(\frac{t - m_1 u}{D_1\sqrt{u}}\right)(1+\overline{o}(1)),$$
$$\mathbf{D}\big[R_u(t) \mid \inf_{0<s\leqslant t} R_u(s) < 0\big] = D_2^2 D_1\sqrt{u}\, g\left(\frac{t - m_1 u}{D_1\sqrt{u}}\right)(1+\overline{o}(1)),$$

as $u \to \infty$.

Remark 1. If $b = c$, where c is from (3), i.e. the premium rate remains unchanged after the first ruin and there is no ruin shock, $m_2(c) = m_2$, where $m_2 = \tau\nu^{1,0}/\nu^{0,1}$.

Remark 2. In the particular case of the Poisson/Exponential model the approximations are easy to express in terms of intensities. Assume that the (i.i.d.)

amounts of claims $\{Y_i\}_{i\geqslant 1}$ and the (i.i.d.) inter-occurrence times $\{T_i\}_{i\geqslant 1}$ are mutually independent and exponential with parameters $\mu > 0$ and $\lambda > 0$ respectively. Then

$$c = \lambda(1+\tau)/\mu \tag{8}$$

and (see e.g., (2.5) – (2.7) in Malinovskii (2000))

$$\begin{gathered}
\varkappa = \mu\tau/(1+\tau), \\
m_1 = \mu/(\lambda\tau(1+\tau)), \quad m_2(b) = b - \lambda\tau/\mu, \\
D_1^2 = 2\mu/(\lambda^2\tau^3) \quad D_2^2 = 2\lambda/\mu^2, \\
C = 1/(1+\tau).
\end{gathered} \tag{9}$$

Evidently, $m_2(c) = \tau\lambda/\mu$. In particular, the approximation for the expectation $\mathbf{E}\big[R_u(t) \mid \inf_{0<s\leqslant t} R_u(s) < 0\big]$ at time point $t = m_1 u$ in this case is

$$\sqrt{2u}g(0)\,/\sqrt{\mu\tau}. \tag{10}$$

Remark 3. The results can be improved in the following directions.

(1) The rate of convergence and corrected approximations as in Malinovskii (1994) can be obtained by more calculations. This is of much interest for theoretical insight and for numerical calculations and will be briefly illustrated in Section 4.

(2) There is still no exact formula for $\psi(w;t,u)$ even in the Poisson/Exponential case (the plausible exact formula for $\psi(w;t,u)$ is a generalization of those of Theorems 2.3 and 2.4 in Malinovskii (2000)).

(3) The approximations of Theorems 1 and 2 can be generalized in the framework of the safety loading τ tending to zero as the initial capital u increases. It is of much interest e.g., for an insurer who develops a competitive strategy which envisages cutting prices. In this framework substantial technical difficulties are generated by a scheme of series. The technique which overcomes these difficulties was developed in Malinovskii (2000).

Remark 4. Straightforward simulation in this context encounters difficulties since the ruin is a rare event even for moderately large u and moderately small τ. It requires much care because the accumulated errors might be significant even for rather accurate random variables generator. The importance sampling method is suggestive in this framework. The accuracy of numerical simulation is a problem of a separate interest.

4. Corrected approximations and a numerical example

For simplicity consider in this section $c = b$. Denote the ladder index $\mathcal{N} = \inf\{n : \overline{S}_n > 0\}$, the ladder height $\mathcal{H} = \overline{S}_{\mathcal{N}}$ and the ladder time point $\mathcal{T} = \overline{U}_{\mathcal{N}}$ and put $\mathcal{W} = \mathbf{E}(\mathcal{T}\mathbf{E}\mathcal{H} - \mathcal{H}\mathbf{E}\mathcal{T})$. Introduce

$$\begin{aligned} &\theta_1 = \frac{\mathbf{E}\mathcal{H}}{1-\mathbf{E}e^{-\varkappa\mathcal{H}}} - \frac{1}{\varkappa}, \quad \theta_2 = \frac{\mathbf{E}\mathcal{T}}{1-\mathbf{E}e^{-\varkappa\mathcal{H}}} - \frac{\mathbf{E}\mathcal{T}e^{-\varkappa\mathcal{H}}}{1-\mathbf{E}e^{-\varkappa\mathcal{H}}}, \\ &\theta_3 = \frac{1}{\varkappa} - \frac{\mathbf{E}\mathcal{H}e^{-\varkappa\mathcal{H}}}{1-\mathbf{E}e^{-\varkappa\mathcal{H}}}, \quad k_1 = \mathbf{E}\mathcal{W}^2, \quad k_2 = \frac{\mathbf{E}\mathcal{W}^3}{6k_1}, \\ &k_3 = \mathbf{E}\mathcal{T}\mathbf{D}\mathcal{H} - \mathbf{E}\mathcal{H}\mathbf{Cov}(\mathcal{H},\mathcal{T}). \end{aligned} \tag{11}$$

The following approximation elaborates the first relation of Theorem 2.

Theorem 3. *Suppose that in the collective risk model with $\tau > 0$ the characteristic function $\beta_{Y,T}(t_1,t_2)$ is absolutely integrable, $0 < D_1, D_2 < \infty$ and $\mathbf{E}T_1^3 < \infty$. Suppose that the premium rate is c as in (3). Then, as $u \to \infty$,*

$$\begin{aligned} &\sup_{t\geqslant 0}\Big|\mathbf{E}\big[R_u(t) \mid \inf_{0<s\leqslant t} R_u(s) < 0\big]\psi(t,u) \\ &- \nu^{1,0}\left(1 - \frac{\nu^{0,2}}{2(\nu^{0,1})^2}\right)\psi(t,u) \\ &- Ce^{-\varkappa u}\tau\frac{\nu^{1,0}}{\nu^{0,1}}D_1\sqrt{u}\left[\left(\frac{t-m_1u}{D_1\sqrt{u}}\right)\Phi_{(0,1)}\left(\frac{t-m_1u}{D_1\sqrt{u}}\right) + \varphi_{(0,1)}\left(\frac{t-m_1u}{D_1\sqrt{u}}\right)\right] \\ &- Ce^{-\varkappa u}\tau\frac{\nu^{1,0}}{\nu^{0,1}}\Phi_{(0,1)}\left(\frac{t-m_1u}{D_1\sqrt{u}}\right)\left(\frac{t-m_1u}{D_1\sqrt{u}}\right)\left(\frac{k_3}{2(\mathbf{E}\mathcal{H})^2} + 3\frac{k_2}{\mathbf{E}\mathcal{H}} - \theta_1\frac{\mathbf{E}\mathcal{T}}{\mathbf{E}\mathcal{H}} + \theta_2\right) \end{aligned}$$

$$- Ce^{-\varkappa u}\tau \frac{\nu^{1,0}}{\nu^{0,1}} \Phi_{(0,1)} \left(\frac{t - m_1 u}{D_1 \sqrt{u}} \right) \left(\theta_2 - \theta_1 \frac{\mathbf{E}\mathcal{T}}{\mathbf{E}\mathcal{H}} - \frac{k_3}{(\mathbf{E}\mathcal{H})^2} - \theta_3 \frac{\mathbf{E}T_1}{\tau \mathbf{E}Y_1} \right)$$
$$- Ce^{-\varkappa u}\tau \frac{\nu^{1,0}}{\nu^{0,1}} \varphi_{(0,1)} \left(\frac{t - m_1 u}{D_1 \sqrt{u}} \right) \left(\frac{t - m_1 u}{D_1 \sqrt{u}} \right) \left(\theta_1 \frac{\mathbf{E}\mathcal{T}}{\mathbf{E}\mathcal{H}} - \theta_2 - 2\frac{k_2}{\mathbf{E}\mathcal{H}} \right) \Bigg| = \overline{o}(e^{-\varkappa u}).$$

Numerical example. Assume that the (i.i.d.) amounts of claims $\{Y_i\}_{i\geqslant 1}$ and the (i.i.d.) inter-occurrence times $\{T_i\}_{i\geqslant 1}$ are mutually independent and exponential with parameters $\mu > 0$ and $\lambda > 0$ respectively.

Lengthy but straightforward calculations similar to those described in Theorem 2 and Lemma 1 of Malinovskii (1994) (see also pp. 890–891 and p. 907 of Malinovskii (2000)) applied to Theorem 3 yield the following approximation for $\mathbf{E}\big[R_u(t) \mid \inf_{0<s\leqslant t} R_u(s) < 0\big]\psi(t,u)$ at the time point $t = m_1 u$ with m_1 from (9):

$$Ce^{-\varkappa u}\Phi_{(0,1)}(0) \left(\frac{\sqrt{2u}}{\sqrt{\mu\tau}} g(0) - \frac{3 + 3\tau + \tau^2}{\mu(1+\tau)} \right). \tag{12}$$

The approximation at the time point $t = m_1 u$ for $\psi(t,u)$,

$$Ce^{-\varkappa u}\Phi_{(0,1)}(0) \left(1 - Q_1(0) \frac{\lambda \tau^{3/2}}{\sqrt{2\mu u}} g(0) \right), \tag{13}$$

where

$$Q_1(0) = \frac{2 + \tau^2}{\lambda\tau(1+\tau)} - \frac{\tau + 2}{2\lambda^2\tau^2},$$

is a corollary of Theorem 1 of Malinovskii (1994). For the conditional expectation $\mathbf{E}\big[R_u(t) \mid \inf_{0<s\leqslant t} R_u(s) < 0\big]$ at the time point $t = m_1 u$ these approximations yield

$$\left(\frac{\sqrt{2u}}{\sqrt{\mu\tau}} g(0) - \frac{3 + 3\tau + \tau^2}{\mu(1+\tau)} \right) \Bigg/ \left(1 - Q_1(0) \frac{\lambda \tau^{3/2}}{\sqrt{2\mu u}} g(0) \right). \tag{14}$$

Compare the approximation (10) and the corrected approximation (14) to the results of direct simulation. For this, simulate N risk reserve trajectories and calculate the mean value of the risk reserve at the time point $t = m_1 u$ over all those trajectories which fall below zero at least once within time $t = m_1 u$. We report the results in the tables below omitting fractional parts. It is seen that the accuracy of (14) appears better than one of (10). For further improvements one has to calculate more terms in the expansions (12) and (13).

TABLE 1: $\lambda = \mu = 1$, $t = 99\,502$, $u = 500$, $\tau = 0.005$, $N = 10\,000$

	Simulation runs							
	1	2	3	4	5	6	7	8
Number of trajectories which fall below zero	287	327	325	315	296	278	286	311
Empirical mean conditioned by ruin	209	224	242	220	214	207	195	222
Approximation (10) for the mean	357							
Corrected approximation (14) for the mean	261							

The data in this table demonstrates a reasonably good accuracy. The poorer accuracy in the following table is due to a smaller τ which brings this case within the scope of the problem mentioned in point (3) of the Remark 3 above. Calculation of more correction terms which are of a smaller order as u grows, but are increasing as τ decreases, becomes here more important.

TABLE 2: $\lambda = \mu = 1$, $t = 499\,500$, $u = 500$, $\tau = 0.001$, $N = 1000$

	Simulation runs							
	1	2	3	4	5	6	7	8
Number of trajectories which fall below zero	189	213	190	222	184	227	396	397
Empirical mean conditioned by ruin	326	369	346	339	368	358	310	335
Approximation (10) for the mean	798							
Corrected approximation (14) for the mean	442							

5. Sketch of the proof

Following von Bahr (1974), introduce the sequence of strict ascending ladder indices

$$\mathcal{N}_1 = \inf\{n : \overline{S}_n > 0\}, \quad \mathcal{N}_k = \inf\left\{n > \mathcal{N}_{k-1} : \overline{S}_n > \overline{S}_{\mathcal{N}_{k-1}}\right\}, \quad k = 2, 3, \dots,$$

the sequence of corresponding ladder heights

$$\mathcal{H}_1 = \overline{S}_{\mathcal{N}_1} = \sum_{i=1}^{\mathcal{N}_1} \overline{X}_i, \quad \mathcal{H}_k = \sum_{i=\mathcal{N}_{k-1}+1}^{\mathcal{N}_k} \overline{X}_i, \quad k = 2, 3, \dots,$$

and the sequence of corresponding ladder time points

$$\mathcal{T}_1 = \overline{U}_{\mathcal{N}_1} = \sum_{i=1}^{\mathcal{N}_1} \overline{T}_i, \quad \mathcal{T}_k = \sum_{i=\mathcal{N}_{k-1}+1}^{\mathcal{N}_k} \overline{T}_i, \quad k = 2, 3, \dots.$$

The following lemma is well known. It follows directly from the fact that $\nu(u)$, the time of the first jump of the sequence $\{\overline{S}_n\}_{n\geqslant 1}$ over the level u, is a ladder index for this sequence.

Lemma 1. *For* $\nu(u) = \inf\{n : \overline{S}_n > u\}$ *and* $\nu_B(u) = \inf\{m : \mathcal{H}_1 + \dots + \mathcal{H}_m > u\}$

$$\nu(u) = \mathcal{N}_{\nu_B(u)}, \quad \overline{S}_{\nu(u)} = \mathcal{H}_1 + \dots + \mathcal{H}_{\nu_B(u)}, \quad \overline{U}_{\nu(u)} = \mathcal{T}_1 + \dots + \mathcal{T}_{\nu_B(u)}.$$

Introduce the density $\mathbf{p}_n(z, x)$ by the equation

$$\mathbf{p}_n(z, x)\, dz\, dx = \mathbf{P}\left(\sum_{i=1}^{n} \mathcal{H}_i \in dz, \sum_{i=1}^{n} \mathcal{T}_i \in dx \right).$$

The following lemma is the key point of the proof. It is a generalization of the representation of the ruin probability $\psi(t, u) = \int_u^\infty e^{-\varkappa z} \int_0^t \mathbf{p}_{\nu_B(u)}(z, x)\, dz\, dx$.

Lemma 2. *For* $\mathbf{p}_{\nu_B(u)}(z, x) = \sum_{n=1}^\infty \int_{v\geqslant 0} \int_{y\geqslant 0} \mathbf{p}_n(u - v, x - y) \mathbf{P}(\mathcal{H}_{n+1} - z + u \in dv, \mathcal{T}_{n+1} \in dy)$ *the equality*

$$\psi(w; t, u) = \int_u^\infty e^{-\varkappa z} \int_0^t \mathbf{p}_{\nu_B(u)}(z, x) \mathbf{P}\big(- V_{N(t-x)} \leqslant w - (z - u) - b(t - x)\big)\, dz\, dx$$

holds true.

The proof of Theorem 1 applies Lemma 2, the Normal approximations with non-uniform bounds for $\mathbf{p}_n(u - v, x - y)$ and the Normal approximation for the distribution of $V_{N(t-x)}$. The latter were considered in Malinovskii (1993).

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