RISK THEORY INSIGHT INTO THE ASSET–LIABILITY AND SOLVENCY ADAPTIVE MANAGEMENT

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Abstract

Bearing in mind the problem of underwriting cycles, two adaptive control strategies regulating the asset–liability balance in the multiperiod controlled risk model composed of chained singleperiodic Poisson–Exponential risk models, are introduced. Solvency performance of these strategies in terms of the probabilities of ruin is analyzed analytically. The strategies are similar, but less sophisticated, than the existent compulsory regulatory procedures.

Keywords: Equity, solvency, multiperiod insurance process, adaptive control, classical risk model.

1. INTRODUCTION

According to Encyclopedia Britannica, "profits in property and liability insurance have tended to rise and fall in fairly regular patterns lasting between five and seven years from peak to peak; this phenomenon is termed the underwriting cycle. Stages of the underwriting cycle may be described as follows: initially, when profits are relatively high, some insurers, wishing to expand sales, start to lower prices and become more lenient in underwriting. This leads to greater underwriting losses. Rising losses and falling prices cause profits to suffer. In the second stage of the cycle, insurers attempt to restore profits by increasing rates and restricting underwriting, offering coverage only to the safest risks. These restrictions may be so severe that insurance in some lines becomes unavailable in the marketplace. Insurers are able to offset a portion of their underwriting loses through earnings on investments. Eventually the increased rates and reduced underwriting losses restore profits. At this point, the underwriting cycle repeats itself.

The general effect of the underwriting cycle on the public is to cause the price of property and liability insurance to rise and fall fairly regularly and to make it more difficult to purchase insurance in some years than in others. The competition among insurers caused by the underwriting cycle tends to create cost bargains in some years. This is especially evident when interest rates are high, because greater underwriting losses will, in part, be offset by greater investment earnings."

Thereby, underwriting cycles are caused by many reasons. One is volatility of financial, investment and other interfacing markets which future development is known incompletely or even unknown at the time of decision making. Other is imperfect management. Third is pure random fluctuations inevitable in the insurance process, even when information is complete and control is optimal. K. Borch (see [**2**], p. 451) observed accordingly that

> "the reserve capital of an insurance company can obviously be considered as a stochastic process, but the laws governing the process will usually be known only partially. As time passes, the company may acquire more knowledge about these laws, for instance by statistical analysis of the current claim payments. The company will then have to decide if, in view of the new knowledge, existing reinsurance arrangements or plans for future dividend payments should be changed. In general the problem of the company will be to devise:
>
> (i) An *information system*: a system for observing the stochastic process as it develops.
>
> (ii) A *decision function*: a set of rules for translating the observations into action."

The classical collective risk theory subject to many technical restrictions seems insufficient to explain the nature of underwriting cycles. The frustration of the initial great hopes was expressed strongly by H. Bohman at the close of his career. Quote from his farewell interview as retiring Chief Editor of the Scandinavian Actuarial Journal (see [**1**], p. 2):

> "I was for a long time deeply involved in this theory, working on the probability of ruin, but I am hesitant over it now . . . From a practical point of view, the theory of collective risk, as initiated by Filip Lundberg, has missed the point, because the underlying model is unrealistic, too simplified. For one thing, a

stationary business should give stationary reserves, as predicted by the control theory."

It must be observed that H. Cramér did justice to F. Lundberg and pointed in [**3**] that problems of that kind were quite clear to the originator of the collective risk theory. Cramér testified that Lundberg was preoccupied much with the adequacy of his model:

> "In view of certain misconceptions that have appeared it is, however, necessary to point out that Lundberg repeatedly emphasizes the practical importance of some arrangement which automatically prevents the risk reserve from growing unduly. This point is, in fact, extensively discussed in the papers of 1909, 1919 and 1926 – 28. One possible arrangement proposed to this end is to work with a security factor $\tau = \tau(x)$ which is a decreasing function of the risk reserve $R(t) = x$. Another possibility is to dispose, at predetermined epochs, of part of the risk reserve for bonus distribution. By either method, the growth of the risk reserve may be efficiently controlled. What Lundberg does in this connection is really to work with a rather refined case of what has much later come to be known as a random walk with two barriers.
>
> From certain quarters, the Lundberg's theory has been declared to be unrealistic because, it is asserted, no limit is imposed on the growth of the risk reserve. In view of what has been said above, it would seem that these critics have not read the author they are criticizing. For a non-Scandinavian author there is, of course, the excuse that most of Lundberg's works are written in Swedish."

In the search of a breakthrough, C. Philipson expressed an opinion shared by many scholars (see [**10**], p. 68):

> "From the development of the classical form [of the risk theory. — V.M.] two lines of development have branched out, one refers to the generalization of the fundamental assumptions... The other refers to the extensions of the decision theory... These lines of development are, however, all based on the fundamental conception of the collective risk theory, which was created by Filip Lundberg..."

Relevance of synthesis of the risk theory and adaptive control theory was emphasized by K. Borch (see [**2**], p. 451):

> "We have now reached the point where the actuarial theory of risk again joins the mainstream of theoretical statistics and applied mathematics. Our general formulation of the actuary's problem leads directly to the general theory of *optimal control processes* or *adaptive control processes*...
>
> The theory of control processes seems to be "tailor–made" for the problems which actuaries have struggled to formulate for more than a century."

Bearing in mind all these premises, the present contribution introduces two adaptive control strategies in the framework of the classical (i.e., Poisson–Exponential with complete information) multiperiodic risk model.

Both strategies are designed to direct or re-direct the year-after-year annual risk reserve into a strip around certain target value corresponding to a prescribed value of the annual probability of ruin, and to keep the actual values of the annual probabilities of ruin within a certain strip around that prescribed value. The former contributes to harmonization of the asset–liability balance (regarding the principle of equity), the later — to the solvency control.

2. MULTIPERIOD INSURANCE PROCESS

A sensible way to amalgamate ideas of the adaptive control and collective risk is to address to multiperiodic controlled model composed of chained singleperiodic risk models (see, e.g., [**8**]). A trajectory of the insurance process with annual accounting and subsequent annual control may be diagramed as

$$\underbrace{w_0 \xrightarrow{\gamma_0} u_0 \xrightarrow{\pi_1} w_1}_{\text{1-st year}} \ \cdots \ \xrightarrow{\pi_{k-1}} \underbrace{w_{k-1} \xrightarrow{\gamma_{k-1}} u_{k-1} \xrightarrow{\pi_k} w_k}_{k\text{-th year}} \ \cdots .$$

According to this diagram (for $k = 1, 2, \dots$), at the end of the $(k-1)$-th year the state variable w_{k-1} is observed; it describes the insurer's position at that time. Then, at the beginning of the k-th year the control variable u_{k-1} is chosen according to the control rule γ_{k-1}, and the k-th year probability mechanism of insurance unfolds. The transition function of this mechanism is denoted by π_k. It defines the insurer's position at the end of the k-th year.

It is noteworthy that a procedure of that type, liable however to critics, is the core of Article 16 [a] in Directives [**5**]. Controlled models, mainly linear, were considered by a range of authors (see e.g., [**4**], [**11**]).

Good asset–liability control, applying both feed-forward and feed-back arguments, has to be based on a fair balance of the principles of equity and solvency. It is a clue to a harmony between the interests of insurer and insured and let the insurance system operate successfully in the long run, i.e., to be solvent. Means to attain and to preserve this balance is therefore a major concern of the insurance administration.

3. TWO ADAPTIVE CONTROL STRATEGIES

Consider a singleperiodic risk model which is called Poisson–Exponential or classical. The annual probability mechanism of insurance comes then from the random variables $\{T_i\}_{i\geqslant 1}$ and $\{Y_i\}_{i\geqslant 1}$, i.i.d. and mutually independent, where T_i are the interclaim times and Y_i are the amounts of claims, exponentially distributed with parameters $\lambda > 0$ and $\mu > 0$. These random variables generate the risk reserve process

$$R_t(u,\tau) = u + ct - \sum_{i=1}^{N(t)} Y_i, \quad t \geqslant 0.$$

Set

$$\boldsymbol{\psi}_t(u,\tau) = \mathsf{P}\{\inf_{0<s\leqslant t} R_s(u,\tau) < 0\}, \quad \boldsymbol{\psi}(u,\tau) = \mathsf{P}\{\inf_{0<s<\infty} R_s(u,\tau) < 0\}$$

for the probability of ruin within time t and for the probability of ultimate ruin, emphasizing dependence on u and $\tau = (c\,\mathsf{E}T_1 - \mathsf{E}Y_1)/\mathsf{E}Y_1 = c\mu/\lambda - 1$ called relative premium loading.

Definition 1. The "target" value $u_{\alpha,t}$ of the risk reserve corresponding to the prescribed value $\alpha \in (0,1)$ of the probability of ruin, is the value $u = u_{\alpha,t}$ which satisfies the equation

$$\boldsymbol{\psi}_t(u;0) = \alpha.$$

Theorem 1. *In the Poisson – Exponential risk model the target value $u_{\alpha,t}$ of the risk reserve corresponding to the prescribed value $\alpha \in (0,1)$ of the probability of ruin* *is the solution of the equation*

$$\int_0^\pi f_t(x; u, 0)\, dx = \pi(1-\alpha), \tag{1}$$

where

$$f_t(x; u, 0) = (2(1-\cos x))^{-1} \exp\big\{(\cos x-1)(u\mu+2\lambda t)\big\}\big[\cos(u\mu \sin x)-\cos(u\mu \sin x+2x)\big].$$

Proof follows from Theorem 2.3 in [**7**], or Remark 2 in [**6**]. Numerical calculations yield the following results.

TABLE 1. Solutions of equation (1) for $\mu = \lambda = 1$.

t	$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.001$	$\alpha = 0.0001$
50	16.7422	27.9872	37.2083	45.3298
100	23.5722	38.6811	50.8686	61.4894
150	28.8077	46.8789	61.3408	73.8748
200	33.2197	53.7878	70.1593	84.3084

Let z be a deviation, either positive or negative, of the past-year-end risk reserve from $u_{\alpha,t}$. Case $z \leqslant 0$ means deficit, case $z > 0$ means surplus. Consider two adaptive control strategies.

3.1. Control without borrowing. Set

$$u_z = u_{\alpha,t} + z \quad \text{and} \quad \tau_{z,t} = -\frac{\mu}{\lambda t} z. \tag{2}$$

This choice of u_z means starting with the initial capital equal to the past-year-end risk reserve, no matter what z is, either a surplus over $u_{\alpha,t}$ or a deficit. The later is reckoned however in the choice of the adaptive loading $\tau_{z,t}$, positive for deficit and negative for surplus.

Bearing in mind that $\mathsf{E}\big(\sum_{i=1}^{N(t)} Y_i\big) = \lambda t/\mu$, for the asset–liability balance one has

$$\mathsf{E}R_t(u_z, \tau_{z,t}) = u_{\alpha,t} \quad \text{for any } z.$$

It means that the control (2) guarantees that the capital of the company at the time t is equal "in the average" to the "target" value $u_{\alpha,t}$.

Introduce

$$\alpha(z) = \frac{\lambda t}{\lambda t - \mu z} \exp\left\{\frac{\mu^2 z\,(u_{\alpha,t}+z)}{\lambda t - \mu z}\right\}$$

and put

$$z^* = \frac{\lambda t}{\mu} - \frac{1}{2\mu} - \frac{\lambda t}{\mu}\Big(1 + \frac{\mu u_{\alpha,t}}{\lambda t} + \frac{1}{4\lambda^2 t^2}\Big)^{1/2}.$$

Applying the expansion $(1+x)^{1/2} = 1 + x/2 + \dots$ in the neighborhood of zero, note that for large t and small $u_{\alpha,t}t^{-1}$

$$z^* = -\frac{u_{\alpha,t}+1}{2} - \frac{1}{8\mu\lambda t} + \dots.$$

THEOREM 2. *For* $z \in [a, b]$, $-u_{\alpha,t} < a < 0 < b < \frac{\lambda}{\mu} t$, *and for the control* (2)

$$\boldsymbol{\psi}_t(u_z, \tau_{z,t}) \leqslant \alpha(z)\mathbf{1}_{\{z<z^*\}} + \alpha(z^*)\mathbf{1}_{\{z>z^*\}}.$$

The upper bound of Theorem 2, which proof contains in [**9**], is illustrated by Fig. 1: z^* is the point where the bowl-shaped upper bound for $\psi_t(u_z, \tau_{z,t})$, i.e., $\psi(u_z, \tau_{z,t})$, achieves its minimum.

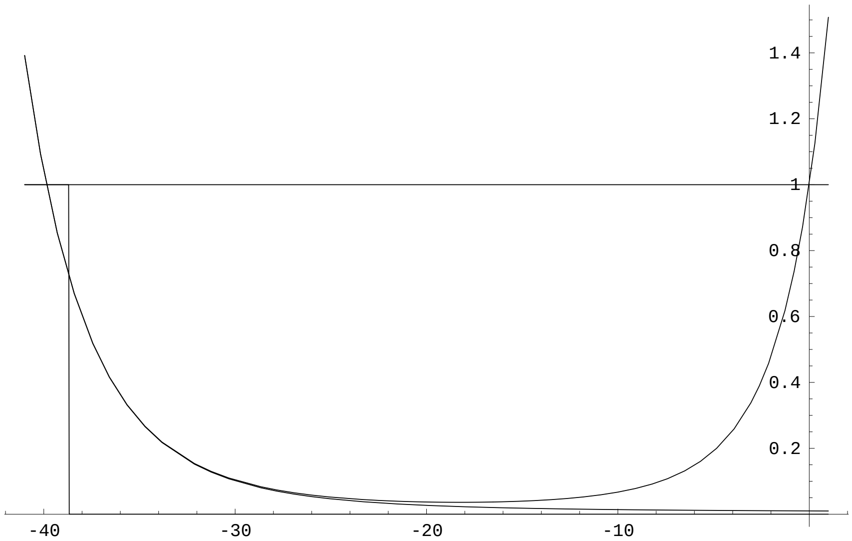

FIGURE 1. Finite time ($t = 200$, $\mu = \lambda = 1$) ruin probability $\psi_t(u_z, \tau_{z,t})$ and its upper bound $\psi(u_z, \tau_{z,t})$ regarded as a functions of z; $u_z = u_{\alpha,t} + z$, $\tau_{z,t} = -\frac{\mu}{\lambda t} z$, target value is $u_{\alpha,t} = 38.6811$ (vertical line shows the value $-u_{\alpha,t}$ bounding the range of z from below) and the prescribed value of the probability of ruin is $\alpha = 0.01$.

3.2. Control with borrowing. Set

$$\bar{u}_z = \begin{cases} u_{\alpha,t}, & z \leqslant 0, \\ u_{\alpha,t} + z, & z > 0 \end{cases} \quad \text{and} \quad \tau_{z,t} = -\frac{\mu}{\lambda t} z. \tag{3}$$

This choice of $\bar{u}_z$ means starting with the initial capital equal to the past-year-end risk reserve when z exceeds $u_{\alpha,t}$, i.e., in case of surplus, and borrowing to make the initial capital equal to $u_{\alpha,t}$, when z is less than $u_{\alpha,t}$, i.e., in case of deficit. The adaptive loading $\tau_{z,t}$ is taken the same as in (2), positive in case of deficit and negative in case of surplus.

Bearing in mind that $\mathsf{E}\big(\sum_{i=1}^{N(t)} Y_i\big) = \lambda t/\mu$, for the asset–liability balance one has

$$\mathsf{E} R_t(\bar{u}_z, \tau_{z,t}) = \begin{cases} u_{\alpha,t} - z, & z \leqslant 0, \\ u_{\alpha,t}, & z > 0. \end{cases}$$

When $z \leqslant 0$ (deficit), the sum $|z|$ must be borrowed at the beginning of the forthcoming insurance year. That sum borrowed amounts exactly to the average surplus at the end of this year; it repays the borrowing.

THEOREM 3. *For $z \in [a, b]$, $-u_{\alpha,t} < a < 0 < b < \frac{\lambda}{\mu} t$, and for the control* (3)

$$\psi_t(\bar{u}_z, \tau_{z,t}) \leqslant \alpha.$$

The assertion of Theorem 3, which proof contains in [**9**], is illustrated by Fig. 2.

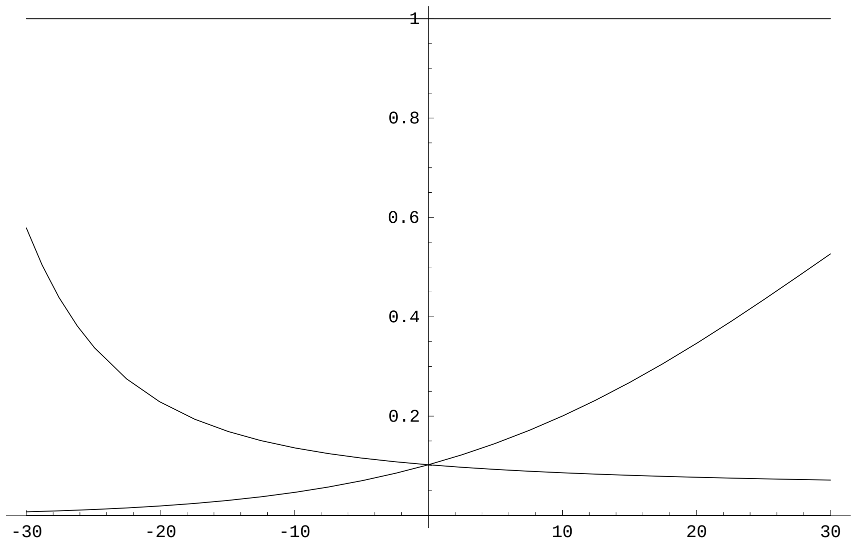

FIGURE 2. Two finite time ($t = 200$, $\mu = \lambda = 1$) ruin probabilities $\boldsymbol{\psi}_t(u_{\alpha,t}, \tau_{z,t})$ (increasing graph) and $\boldsymbol{\psi}_t(u_{\alpha,t} + z, \tau_{z,t})$ (decreasing graph) regarded as functions of z; $\tau_{z,t} = -\frac{\mu}{\lambda t} z$, target value is $u_{\alpha,t} = 33.2197$, the prescribed value of the probability of ruin is $\alpha = 0.1$.

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